Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 177 (2007) 1610–1625
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

# Randomly generating portfolio-selection covariance matrices with specified distributional characteristics

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Available online 29 November 2005

**Abstract**

In portfolio selection, there is often the need for procedures to generate "realistic" covariance matrices for security returns, for example to test and benchmark optimization algorithms. For application in portfolio optimization, such a procedure should allow the entries in the matrices to have distributional characteristics which we would consider "realistic" for security returns. Deriving motivation from the fact that a covariance matrix can be viewed as stemming from a matrix of factor loadings, a procedure is developed for the random generation of covariance matrices (a) whose off-diagonal (covariance) entries possess a pre-specified expected value and standard deviation and (b) whose main diagonal (variance) entries possess a likely different pre-specified expected value and standard deviation. The paper concludes with a discussion about the futility one would likely encounter if one simply tried to invent a valid covariance matrix in the absence of a procedure such as in this paper.


*Keywords:* Random covariance matrices; Random correlation matrices; Positive semidefinite matrices; Covariance matrix factorization; Portfolio selection; Portfolio optimization

## 1. Introduction

In finance there is the well-known problem of portfolio selection. The standard problem of portfolio selection is described as follows in which we see the central role played by a covariance matrix. Assume

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[1] Research conducted while a Visiting Scholar at the Department of Banking and Finance, Terry College of Business, University of Georgia, October 2003–March 2004.

0377-2217/$ - see front matter 
doi:10.1016/j.ejor.2005.10.014

(a) the beginning of a holding period,
(b) the end of the holding period,
(c) an initial sum to be invested,
(d) $n$ securities whose end-of-holding-period returns are given by the random vector $\mathbf{r} = (r_1, \ldots, r_n)$ with expected value $\mu = (\mu_1, \ldots, \mu_n)$ and $n \times n$ covariance matrix.

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & & \\ \vdots & & & \vdots \\ \sigma_{n1} & & \cdots & \sigma_{nn} \end{bmatrix}.$$

In $\Sigma$, each diagonal element $\sigma_{ii}$ is a *variance* (variance of random variable $r_i$) and each off-diagonal element $\sigma_{ij}$, $i \neq j$, is a *covariance* (covariance of random variables $r_i$ and $r_j$). We observe that $\Sigma$ is symmetric as $\sigma_{ij} = \sigma_{ji}$ for all $i, j$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the vector of investment proportion weights. With the monetary payoff (portfolio return) for the portfolio defined by $\mathbf{x}$ being

$$\sum_{i=1}^{n} r_i x_i$$

the difficulty in portfolio selection is that the $r_i$ are not known until the end of the holding period, but the $x_i$ must be chosen at the beginning of the holding period. Under the supposition that investors pursue portfolio expected return $\mu^{\mathrm{T}}\mathbf{x}$ while wishing to avoid portfolio variance $\mathbf{x}^{\mathrm{T}}\Sigma\mathbf{x}$, we have the bi-criterion "risk-return" portfolio selection formulation of Markowitz [28–30] and Markowitz and Todd [31]

$$\min \quad \{\mathbf{x}^{\mathrm{T}}\Sigma\mathbf{x}\} \tag{1.1}$$

$$\max \quad \{\mu^{\mathrm{T}}\mathbf{x}\}, \tag{1.2}$$

$$\text{s.t.} \quad \mathbf{x} \in S, \tag{1.3}$$

where $S$ is the region of all feasible investment proportion vectors.

While (1.1)–(1.3) is at the heart of "modern portfolio theory" [11], a growing interest has been observed in multiple objective quadratic-linear programming [23,16] and in extending approaches to portfolio selection to accommodate investors who might have additional criterion concerns such as dividends, liquidity, number of securities in the portfolio, social responsibility, amount invested in R&D, and so forth [1–4,6,10,12,19,20,22,33,36,38,26,37,40].

The diversity of above multiple objective quadratic-linear programming and multiple criteria portfolio-selection approaches has aggravated a long-standing incapability. There is a reason, despite the legions of papers written on portfolio selection over the past five decades, that there is a dearth of computational results papers on portfolio optimization. Just try to find a paper that reports on the time it takes to solve portfolio-selection problems of different sizes and characteristics. We are unaware of any good references on this. The problem is the covariance matrix. Except in small instances, one cannot simply create covariance matrices whose elements have any resemblance to those seen in portfolio selection by assigning random numbers. The difficulty is that for a square matrix to be a valid covariance matrix, it must be *positive semidefinite*, where

**Definition 1.** Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then $\mathbf{A}$ is *positive semidefinite* if and only if $\mathbf{A}$ is symmetric and

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \geqslant 0$$

for all $\mathbf{x} \in \mathbb{R}^n$.

If attempting to make up, without a procedure, a portfolio-selection realistic covariance matrix up to size $10 \times 10$, one might have reasonably good luck, but the success rate quickly declines as $n$ gets larger. For instance, in the experiments discussed in Section 8, we were not able to generate a single valid $50 \times 50$ covariance matrix by assigning random numbers in 800 tries. This is serious because covariance matrices in portfolio selection are often not small. Sizes of $1000 \times 1000$ are not uncommon.

If this is the case, then how has conventional portfolio optimization managed to survive over the years? Apart from obtaining simplistic covariance matrices from averaging techniques as described in Chapter 8 of Elton et al. [11], the field has long been resigned to the often messy process, since there has been no other way, of generating its necessary covariance matrices from historical data. While this may come with the territory on applications, this hardly meets the needs of research when all manner of covariance matrices may be needed for experimenting with, testing, comparing and benchmarking various strategies, approaches and algorithms.

Four difficulties can be expected when attempting to obtain covariance matrices of meaningful size from historical data. One is that access to a sufficient database (data universe) must be obtained, often very expensive. Two, seemingly almost always with missing or defective data and in a format not the way you want it, the dataset typically must be "cleaned up", a time consuming process. Three, you have no say about the "characteristics" of the covariance matrix that results, you get what you get.

By *characteristics* we mean the distributional characteristics of the variances (diagonal elements) and covariances (off-diagonal elements) in a $\Sigma$. For example, from monthly data for 200 securities randomly selected from the S&P Composite 1500, Fig. 1 shows how the diagonal variance elements and off-diagonal covariance elements are distributed in the covariance matrix derived from the data. The taller of the distributions (thin line), with mean of 0.00245 and standard deviation 0.00330, is that of the 19,900 unique covariances. The other distribution (thick line), with mean of .01860 and standard deviation .01699, is that of the 200 variances.

A fourth encountered difficulty is that after a historical covariance matrix is obtained, one has no way to generate additional different, but similar in terms of characteristics, test-problem covariance matrices for computational testing. Having access to only one or a small number of data universes runs the risk of researchers optimizing their work around only the datasets they possess. It has been our experience in Hirschberger, Qi and Steuer [16,17], at least for purposes of computational testing, that the situation is dramatically improved if one is in possession of a random covariance matrix generation capability. Then one can produce, on demand, strings of valid covariances matrices of any length, of any size, and of any (mathematically allowable) diagonal/off-diagonal distributional characteristics. In this way, it is almost as if, at least for computing covariance matrices, we have an infinite number of data universes at our disposal.

With this as background, the purpose of the paper is to present a procedure for not only the random generation of valid covariance matrices of any size, but for the random generation of covariance matrices

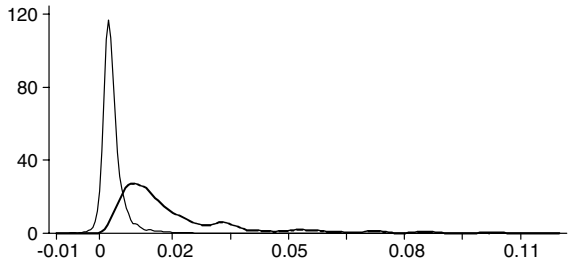


Fig. 1. Distributions of the covariances (thin line) and variances (thick line) of 200 randomly selected securities from the S&P Composite 1500.

whose diagonal elements possess a pre-specified expected value and standard deviation and whose off-diagonal elements possess a likely different pre-specified expected value and standard deviation. For instance, if one were in need of a series of randomly generated covariance matrices whose variances and covariances have distributions as in Fig. 1, the procedure could do so quite closely in each instance. In this way, this paper should provide researchers in quadratic-linear programming and portfolio selection with a robust tool for test problem creation which should be of assistance in many testing, stress testing, computational comparison, or benchmarking situations.

It is difficult to do a literature review for a paper like this since we are unaware of the reporting of any other research directed at the random generation of covariance matrices to match pre-defined diagonal/off-diagonal distributional characteristics. Nevertheless, over the last 35 years, there has been a sprinkling of papers on the construction, generation, and use of random correlation matrices, mostly in other disciplines, and in all cases, for different purposes. Not surprisingly, none of them have been of direct assistance to us in this paper, but we would probably be remiss in not briefly mentioning them since they represent the most nearby literature.

From the beginning, most papers in the sprinkling have focused on the random generation of correlation matrices to match given sets of eigenvalues to meet application needs predominately in numerical statistical analysis (factor analysis, stepwise regression, principal components analysis, and so forth) and signal processing. Papers here include those by Tucker et al. [39], Bendel and Mickey [5], Davies and Higham [9], Holmes [18], Lin and Bendel [27], and Marsaglia and Olkin [32].

Striking closer to home, there have been some random correlation and covariance matrix contributions with linkages to finance, but still not the same as this paper. One is by Chopra and Ziemba [8] in which the elements of a covariance matrix are randomly perturbed to test the effects of changes in the elements on optimal portfolio choice. With a different take on covariance matrices, there are the papers by Laloux et al. [24,25], Pafka and Kondor [34] and Pafka et al. [35] in which they use results from the theory of random matrices to reduce noise in empirical covariance matrices.

Finally, from the literature of matrix nearness problems (surveyed by Higham [13]), there is the recent paper by Higham [14]. Motivated by a mutual fund situation, the purpose of his paper is to compute from the cone of positive semidefinite matrices the matrix that is nearest to a matrix not in the cone. This is potentially very useful in portfolio optimization with correlation matrices in which there may be inadvertent errors or inconsistencies that must be ironed out. Perhaps it may be possible to combine Higham's paper and this paper in some kind of complementary fashion in future research.

While all but Chopra and Ziemba [8] have been about correlation matrices, this is equivalent to all having been about covariance matrices. This is because, when all standard deviations $\sigma_i = \sqrt{\sigma_{ii}}$ are nonzero, associated correlation and covariance matrices, $C$ and $\Sigma$, respectively, are simply re-scaled versions of one another, i.e.,

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) C \operatorname{diag}(\sigma_1, \ldots, \sigma_n),$$
$$C = \operatorname{diag}\left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n}\right) \Sigma \operatorname{diag}\left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n}\right).$$

The paper proceeds as follows. With covariance matrix factorization as a reference point, the strategy of the paper is introduced in Section 2. The theory behind the procedure by which covariance matrices can be randomly generated to specification is developed over Sections 3–5. Three different modes of operation for the procedure are outlined in Section 6. Examples of the procedure applied to four problems in which $n = 50, 200, 500$ and 1000 are overviewed in Section 7. Experience reporting on the futility likely to be encountered if one simply tried to "guess" a covariance matrix without the aid of a procedure concludes the paper in Section 8.

## 2. Covariance matrix factorization

Covariance matrix factorization is sometimes employed in financial research to randomly generate hypothetical return vectors $\mathbf{r} \in \mathbb{R}^n$ from a given $\Sigma \in \mathbb{R}^{n\times n}$. From the previous section we know that all covariance matrices are positive semidefinite, and from linear algebra we know that each positive semidefinite matrix of order $n$ has a *root matrix* $L \in \mathbb{R}^{n\times p}, p \leqslant n$, such that $n \times n$ matrix $LL^{\mathrm{T}}$ is the positive semidefinite matrix in question. In this way, each covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$ has a root matrix $L \in \mathbb{R}^{n\times p}$ such that

$$\Sigma = LL^{\mathrm{T}}. \tag{2}$$

According to the theory of factor analysis, $L$ can be interpreted as a matrix of factor loadings for some vector of risk factors $\mathbf{y} \in \mathbb{R}^p$ as follows:

$$\mathbf{r} = \mu + L\mathbf{y}. \tag{3}$$

With respect to (3), we note two things. One is that for each $\mathbf{y}$-vector randomly drawn from the standard multinormal distribution, (3) generates a random return vector $\mathbf{r}$ with expected value $\mu$ and covariance matrix given by $LL^{\mathrm{T}}$. Of course, if we were to re-use a series of $\mathbf{r}$-vectors generated in this way to compute a sample covariance matrix, we would obtain only an estimate $\widehat{\Sigma}$ of the original covariance matrix $\Sigma = LL^{\mathrm{T}}$.

As shown in Bratley et al. [7], the other involves a collection $Y$ of $q \geqslant n + 1$ specially constructed $\mathbf{y}$-vectors that have exactly $\mathbf{1} \in \mathbb{R}^p$ as mean and exactly identity matrix of order $p$ as covariance matrix. Let $M$ be $n \times q$ whose columns are each $\mu$. Then expression $M + LY$ yields $q$ return vectors $\mathbf{r} \in \mathbb{R}^n$ of exact mean $\mu$ and exact covariance matrix $\Sigma$.

While the above is interesting because, as we shall see in the next paragraph, it provides insight, the point of this paper is *not* the random generation of return vectors. Rather, the purpose of this paper is to randomly generate different instances of covariance matrices with pre-chosen diagonal and off-diagonal distributional characteristics. However, with random return vectors as background, the capability of this paper can be viewed as completing the picture with regard to both the random generation of return vectors and the random generation of covariance matrices.

While (2) true, its converse is more general. That is, for any matrix $F \in \mathbb{R}^{n\times m}$ where $m \geqslant 1$, $n \times n$ matrix $FF^{\mathrm{T}}$ is positive semidefinite. This means that a random covariance matrix can be obtained by creating a random $F$-matrix first. In this way we see the strategy of the paper. It is to develop a method for randomly generating the $f_{ij}$ elements of an $F$ matrix such that the resulting positive semidefinite matrix (i.e., covariance matrix) has the diagonal and off-diagonal element distributional characteristics desired.

To accomplish the goals of this paper of holding the diagonal and off-diagonal elements of a $\Sigma$ to certain distributional requirements, all elements $f_{ij}$ of the $F$-matrices constructed in this paper are composed of independent and identically distributed random components.

## 3. Procedure inputs

To be clear on notation, for a random variable $X$, let $E[X]$ denote expected value, and let variance, standard deviation, skewness, and kurtosis be defined as follows:

$$V[X] = E[(X - E[X])^2],$$

$$\mathrm{Std}[X] = V[X]^{1/2},$$

$$\mathrm{Sk}[X] = E[(X - E[X])^3], \tag{4}$$

$$K[X] = E[(X - E[X])^4]. \tag{5}$$

Note that in (4) and (5), for simplicity in this paper, we will not be normalizing skewness by $V[X]^{3/2}$ and kurtosis by $V[X]^2$.

To specify the desired control over the diagonal and off-diagonal elements of a covariance matrix to be generated, the procedure accepts four (sometimes five) quantities as inputs. Two are $e$ and $\sqrt{v}$, the expected value and standard deviation of the off-diagonal element covariances

$$e := E[\sigma_{ij}] \quad \sqrt{v} := \mathrm{Std}[\sigma_{ij}] \quad \text{over all pairs } i, j, \ i \neq j,$$

where for the proposed procedure to work, it is required that $e > 0$. This is not considered a severe restriction as the vast majority of securities exhibit positive comovements with one another. Two other inputs, $\bar{e}$ and $\sqrt{\bar{v}}$, are the expected value and standard deviation of the diagonal-element variances

$$\bar{e} := E[\sigma_{ii}] \quad \sqrt{\bar{v}} := \mathrm{Std}[\sigma_{ii}] \quad \text{for all } i.$$

Here it is required that $\bar{e} \geqslant e$. This is a requirement because it is a property of covariance matrices, proved in Appendix A, that the average of the off-diagonal elements of a covariance matrix cannot be greater than the average of the diagonal elements.

Depending on conditions, a fifth input $s$, reflecting the skewness of the off-diagonal covariances, where

$$s := \mathrm{Sk}[\sigma_{ij}] \quad \text{over all pairs } i, j, \ i \neq j$$

is sometimes possible.

Since for every covariance matrix $\Sigma$ there exists an $F$-matrix of some horizontal dimension $m$ such that $\Sigma = FF^{\mathrm{T}}$, the goal of this paper is to utilize the above inputs to construct the $f_{ij}$ elements of an $F$ in such a way that matrix multiplication $FF^{\mathrm{T}}$ results in a $\Sigma$ possessing the desired diagonal and off-diagonal distributional characteristics. In other words, the procedure of this paper shows how the inputted $e, v, \bar{e}, \bar{v}$ (and sometimes $s$) are transformed into $\hat{e}, \hat{v}, \hat{s}, \hat{k}$ and $m$ values where

$$E[f_{ij}] := \hat{e} \quad V[f_{ij}] := \hat{v} \quad \mathrm{Sk}[f_{ij}] := \hat{s} \quad K[f_{ij}] := \hat{k} \quad \text{over all pairs } i, j, \ i = 1, \ldots, n \ j = 1, \ldots, m$$

such that when drawing the $f_{ij}$ values from a distribution with these characteristics, the entries in the resulting covariance matrix reflect the distributional structure specified by the four or five inputs.

## 4. Theory of the $f_{ij}$ for the computation of covariances

To develop the $\hat{e}, \hat{v}$ and $\hat{s}$ moments of the distribution from which the $f_{ij}$ are drawn so as to generate the off-diagonal elements of a covariance matrix that have expected value $e$ and standard deviation $\sqrt{v}$, we have Theorem 1.

**Theorem 1.** *Let the $f_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, be independent random variables, identically distributed by some probability distribution $\mathfrak{F}$. Then the (off-diagonal) covariance elements $\sigma_{ij}$, $i, j = 1, \ldots, n$, $i \neq j$, in $FF^T$ have expected value $e \geqslant 0$, variance $v \geqslant 0$ and skewness $s > 0$ if and only if*

$$\hat{e} = \sqrt{\frac{e}{m}}, \tag{6}$$

$$\hat{v} = -\hat{e}^2 + \sqrt{\hat{e}^4 + \frac{v}{m}}, \tag{7}$$

$$\hat{s} = -\hat{e}^3 - 3\hat{e}\hat{v} + \sqrt{3\hat{e}^2\hat{v}^2 + 6\hat{e}^4\hat{v} + \hat{e}^6 + \frac{s}{m}}, \tag{8}$$

*where formula (8) is valid if and only if*

$$s \geqslant -\frac{1}{m}\left[\frac{e^3}{m} + 3ve\right]. \tag{9}$$

The right-hand side of (9) specifies a lower bound on an inputted value for the skewness $s$ of the covariances. If $s$ is not specified, then $\hat{s}$ will be determined later on. Since randomly generated covariances are to have positive expected values, the lower bound is negative which is hardly a restriction since covariances in practice tend to exhibit mild positive skewness.

**Proof.** In the following, we will make use of the well-known relations:

$$E[(X - E[X])^2] = E[X^2] - E[X]^2, \tag{10}$$

$$E[(X - E[X])^3] = E[X^3] - 3E[X^2]E[X] + 2E[X]^3, \tag{11}$$

which hold for any random variable $X$ for which the necessary moments exist. Note that (10) and (11) can be expressed as

$$E[X^2] = V[X] + E[X]^2, \tag{12}$$

$$E[X^3] = \mathrm{Sk}[X] + 3V[X]E[X] + E[X]^3. \tag{13}$$

Throughout the proof, let $Y$ and $Z$ be independent random variables distributed according to $\mathfrak{F}$. □

**Proof of formula (6).** By definition, $\sigma_{ij} = f_{i1}f_{j1} + \cdots + f_{im}f_{jm}$. By the additivity of expected values

$$E[\sigma_{ij}] = E[f_{i1}f_{j1}] + \cdots + E[f_{im}f_{jm}].$$

Since the $f_{ik}$ and $f_{jk}$ are independent for $i \neq j$, we obtain

$$e = E[\sigma_{ij}] = E[f_{i1}]E[f_{j1}] + \cdots + E[f_{im}]E[f_{jm}] = m\hat{e}^2.$$

Upon rearrangement, this establishes (6). □

**Proof of formula (7).** Since the products $f_{il_1}f_{jl_1}$ and $f_{il_2}f_{jl_2}$ are independent for $i \neq j$, for the variance of the covariances we have

$$v = V[\sigma_{ij}] = V[f_{i1}f_{j1}] + \cdots + V[f_{im}f_{jm}] = mV[YZ].$$

By (10) and by the independence of $Y$ and $Z$, we have

$$v = m(E[Y^2]E[Z^2] - E[Y]^2E[Z]^2).$$

Applying (12) to $Y$ and $Z$ yields

$$v = m(V[Y]V[Z] + V[Y]E[Z]^2 + V[Z]E[Y]^2 + E[Y]^2E[Z]^2 - E[Y]^2E[Z]^2),$$

which in turn yields

$$v = m(\hat{v}^2 + 2\hat{v}\hat{e}^2).$$

Rearranging into a recognizable quadratic form

$$\hat{v}^2 + 2\hat{v}\hat{e}^2 - \frac{v}{m} = 0$$

we observe two things. One is that the discriminant $4\hat{e}^4 + \frac{4v}{m}$ is nonnegative, and the other is that $\sqrt{\hat{e}^4 + \frac{v}{m}} \geqslant \hat{e}^2$. This means that there is only one nonnegative solution for $\hat{v}$ and it can be found in the positive branch of the quadratic formula which then yields (7). □

**Proof of formula** ((8) *and* (9)). By (11) and (12), we can express the third central moment of $\sigma_{ij}$ as

$$s = E[(\sigma_{ij} - E[\sigma_{ij}])^3] = E[\sigma_{ij}^3] - 3E[\sigma_{ij}^2]E[\sigma_{ij}] + 2E[\sigma_{ij}]^3 = E[\sigma_{ij}^3] - 3(V[\sigma_{ij}] + E[\sigma_{ij}]^2)E[\sigma_{ij}] + 2E[\sigma_{ij}]^3.$$

Since $E[\sigma_{ij}] = e = m\hat{e}^2$ and $V[\sigma_{ij}] = v = m(\hat{v}^2 + 2\hat{v}\hat{e}^2)$, this is equivalent to

$$s = E[\sigma_{ij}^3] - 3m^2\hat{e}^2\hat{v}^2 - 6m^2\hat{e}^4\hat{v} - m^3\hat{e}^6$$

and only $E[\sigma_{ij}^3]$ remains to be computed. By definition of $\sigma_{ij}$,

$$E[\sigma_{ij}^3] = E\left[\sum_{l=1}^{m}(f_{il}f_{jl})^3 + 3\sum_{\substack{l_1,l_2=1\\ l_1\neq l_2}}^{m}(f_{il_1}f_{jl_1})(f_{il_2}f_{jl_2})^2 + 6\sum_{\substack{l_1,l_2,l_3=1\\ l_1<l_2<l_3}}^{m}(f_{il_1}f_{jl_1})(f_{il_2}f_{jl_2})(f_{il_3}f_{jl_3})\right].$$

Since each $E[(f_{il}f_{jl})^a] = E[(YZ)^a]$ regardless of the $a \in \mathbb{N}$, $i \neq j$, and $l$, we can apply, when $m > 2$, the well-known formulas

$$\sum_{\substack{l_1,l_2=1\\ l_1\neq l_2}}^{m} 1 = m(m-1) \qquad \sum_{\substack{l_1,l_2,l_3=1\\ l_1<l_2<l_3}}^{m} 1 = \frac{1}{6}m(m-1)(m-2).$$

Observing that $\frac{1}{6}m(m-1)(m-2) = \frac{m}{6}(m^2 - 3m + 2)$, we write

$$E[\sigma_{ij}^3] = mE[(YZ)^3] + 3m(m-1)E[YZ]E[(YZ)^2] + m(m^2 - 3m + 2)E[YZ]^3.$$

More precisely, since $Y$ and $Z$ are independent

$$E[\sigma_{ij}^3] = mE[Y^3]^2 + 3m(m-1)E[Y]^2E[Y^2]^2 + m(m^2 - 3m + 2)E[Y]^6.$$

By (12) and (13), we know that $E[Y^2] = \hat{v} + \hat{e}^2$ and $E[Y^3] = \hat{s} + 3\hat{v}\hat{e} + \hat{e}^3$. Thus

$$E[\sigma_{ij}^3] = m\{\hat{s}^2 + \hat{s}(6\hat{v}\hat{e} + 2\hat{e}^3) + (3m+6)\hat{v}^2\hat{e}^2 + 6m\hat{v}\hat{e}^4 + m^2\hat{e}^6\}.$$

Inserting this relation into our formula for $s$ yields

$$s = m(\hat{s}^2 + \hat{s}(6\hat{v}\hat{e} + 2\hat{e}^3) + 6\hat{v}^2\hat{e}^2).$$

Rearranging into a recognizable quadratic form, we have

$$\hat{s}^2 + \hat{s}(6\hat{v}\hat{e} + 2\hat{e}^3) + 6\hat{v}^2\hat{e}^2 - \frac{s}{m} = 0.$$

If this equation is solvable, the positive branch of the quadratic formula yields the solution for $\hat{s}$ as given by formula (8). However, for this equation to be solvable, the discriminant has to be nonnegative, i.e.,

$$s \geqslant -m(3\hat{v}^2\hat{e}^2 + 6\hat{v}\hat{e}^4 + \hat{e}^6). \tag{14}$$

After inserting (6) and (7) into (14), formula (9) results. □

## 5. Theory of the $f_{ij}$ for the computation of variances

To determine $m$, the number of columns in $F$, and the fourth central moment $\hat{k}$ of the distribution from which the $f_{ij}$ are to be drawn so that the diagonal elements of a covariance matrix have expected value $\bar{e}$ and standard deviation $\sqrt{\bar{v}}$, we have Theorem 2.

**Theorem 2.** *Under the assumptions of Theorem 1, the (diagonal-element) variances $\sigma_{ii}$, $i = 1, \ldots, n$, in the $FF^T$ have expected value $\bar{e} \geqslant 0$ and variance $\bar{v} \geqslant 0$ if and only if*

$$\bar{e} = m(\hat{v} + \hat{e}^2), \tag{15}$$

$$\bar{v} = m(\hat{k} + 4\hat{s}\hat{e} - \hat{v}^2 + 4\hat{v}\hat{e}^2). \tag{16}$$

**Proof.** We use the well-known relation

$$E[(X - E[X])^4] = E[X^4] - 4E[X^3]E[X] + 6E[X^2]E[X]^2 - 3E[X]^4, \tag{17}$$

which holds for any random variable $X$ for which the necessary moments exist. Note that by combining this relation with (10) and (11), we receive the equivalent formulation

$$E[X^4] = K[X] + 4\mathrm{Sk}[X]E[X] + 6V[X]E[X]^2 + E[X]^4. \tag{18}$$

Throughout the proof, let $Y$ be a random variable distributed according to $\mathfrak{F}$. □

**Proof of formula (15).** By the additivity of the expectation

$$\bar{e} = E[\sigma_{ii}] = E[f_{i1}^2] + \cdots + E[f_{im}^2] = mE[Y^2].$$

Then relation (12) yields (15). □

**Proof of formula (16).** Since $f_{il_1}^2$ and $f_{il_2}^2$ are independent for $l_1 \neq l_2$, the variances are additive, such that

$$\bar{v} = V[\sigma_{ii}] = V[f_{i1}^2] + \cdots + V[f_{im}^2] = mV[Y^2].$$

Applying relation (10), we receive

$$\bar{v} = m(E[Y^4] - E[Y^2]^2) = m(E[Y^4] - (V[Y] + E[Y]^2)^2)$$

and subsequently applying formula (18) yields

$$\bar{v} = m(K[Y] + 4\mathrm{Sk}[Y]E[Y] - V[Y]^2 + 4V[Y]E[Y]^2).$$

Now (16) is immediate. □

Re-expressing (15) in terms of original inputs, which is shown in Appendix B, we have

$$m = \frac{\bar{e}^2 - e^2}{v}. \tag{19}$$

This means that we can always approximately achieve inputted values $e$, $v$ and $\bar{e}$ by rounding $m$ to the nearest integer. That is, by rounding, $m$ changes by at most 0.5. From Eqs. (15) and (16), it is clear that there is at most an error of 1% in $\bar{e}$ and $\bar{v}$ if we assume $m = 50$ (which is close to the $m$-values seen in the simulations of Table 2). We also see that $m$ is independent of $n$ being mostly dependent on the difference between $\bar{e}$ and $e$. Note that we do not have to worry about $m$ being negative, because as commented on in Section 3, $\bar{e}$ is not to be specified less than $e$. In one final matter, suppose we were to obtain a negative value for $\hat{k}$ from (16). Re-writing (16) as

$$\hat{k} = \frac{\bar{v}}{m} + \hat{v}^2 - 4\hat{s}\hat{e} - 4\hat{v}\hat{e}^2 \tag{20}$$

we see that a negative $\hat{k}$ can always be driven into positive territory by suitably lowering $\hat{s}$. This has not been necessary for all the experiments we conducted with parameters derived from market data.

## 6. Choice of distribution function for the $f_{ij}$

The method for randomly generating covariance matrices of this paper can be applied in several modes. We will assume that $m$ is always adjusted to accommodate inputted expected value $\bar{e}$. In addition, one may wish to

*Mode ev* $-\ \bar{e}$*:* Only replicate the expected value $e$ and standard deviation $\sqrt{v}$ of the off-diagonal elements of $\Sigma$.
*Mode ev* $-\ \bar{e}\bar{v}$*:* Replicate the expected value and standard deviation of the off-diagonal elements, and the standard deviation $\sqrt{\bar{v}}$ of the diagonal elements of $\Sigma$.
*Mode evs* $-\ \bar{e}\bar{v}$*:* Replicate the expected value, standard deviation and skewness $s$ of the off-diagonal elements, and the standard deviation of the diagonal elements of $\Sigma$.

For each mode, different families of distribution functions may be applicable for the drawing of the $f_{ij}$. For instance, for a family to be applicable, there must be algorithms to fit to the distribution the first 2 to 4 central moments. Also, there must exist closed-form formulas or at least good approximations for the corresponding quantile (inverse distribution) functions. This is the case for the normal, lognormal and Johnson [21] families of distributions. However, we wish to point out that not only might the values of the elements in the matrix being randomly generated differ depending on the choice of distribution function, but the choice of distribution function might well affect the interdependences among the elements.

### 6.1. Mode ev $-\ \bar{e}$

The mode is always possible, and can easily and effectively be implemented using the normal distribution since only the moments $\hat{e}$ and $\hat{v}$ are required to replicate $e$, $v$ and $\bar{e}$. It is also possible to use the uniform distribution, but we have found the kurtosis of this distribution to be too low to produce a satisfactory variance among the diagonal elements. Before starting, we compute the $m$, $\hat{e}$ and $\hat{v}$ according to formulas (19), (6) and (7). Then the procedure is

(a) Generate standard normally distributed variables $q_{ij}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$.
(b) Let $f_{ij} = \hat{e} + \sqrt{\hat{v}}q_{ij}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$.
(c) Compute $\Sigma = FF^{\mathrm{T}}$.

### 6.2. Mode ev $-\ \bar{e}\bar{v}$

This mode requires a more flexible distribution family since after computing $m$, $\hat{e}$ and $\hat{v}$, moments $\hat{s}$ and $\hat{k}$ must be determined to satisfy Eq. (16). Theoretically, this mode is always possible using distributions from the Johnson family, which includes the lognormal distribution. But in fact, most cases may be covered by the lognormal distribution only. Since it is much easier to work with the lognormal than the Johnson family, we describe a procedure to compute $\hat{s}$ and $\hat{k}$ for the lognormal distribution.

To solve for $\hat{k}$ and $\hat{s}$ under the lognormal in our situation, two conditions must be satisfied. One is that under the lognormal, $\hat{k}$ and $\hat{s}$ are related through $\omega$ as described in Johnson [21] and Hill et al., [15] as follows:

$$\frac{\hat{k}(\omega)}{\hat{v}^2} = (\omega^4 + 2\omega^3 + 3\omega^2 - 3) \quad \frac{\hat{s}(\omega)}{\hat{v}^{3/2}} = \pm\sqrt{(\omega-1)(\omega+2)^2} \tag{21}$$

for $\omega \geqslant 1$. Parameterizing $\omega$ from $1 \to +\infty$, points

$$\left(\frac{\hat{k}(\omega)}{\hat{v}^2}, \frac{\hat{s}(\omega)}{\hat{v}^{3/2}}\right)$$

trace the bullet-shaped curve in Fig. 2. The points start at the minimum normalized kurtosis point $(3, 0)$ when $\omega = 1$, and proceed above and below the horizontal axis (as a result of the $\pm$ in (21)) to the right as $\omega$ increases. The other is condition (20) which plots as the negatively sloped dashed line in the Fig. 2.

Provided that (20) intersects the horizontal axis to the right of 3 (highly likely), the points of intersection between the bullet-shaped curve and the dashed line can be found by solving

$$\hat{k}(\omega) + 4\hat{e}\hat{s}(\omega) + \left[-\hat{v}^2 + 4\hat{v}\hat{e}^2 - \frac{\bar{v}}{m}\right] = 0$$

for its two roots. In this paper we opt for the solution on the upper half of the bullet-shaped curve (i.e., the one with the smallest $\omega$). It appears that this root is more in line with portfolio optimization as it has positive as opposed to negative skewness and more kurtosis (fatter tails).

With $\omega$ now determined, we can proceed as described in Johnson [21] and Hill et al. [15] with the other parameters of the lognormal as follows:

$$\delta = (\ln \omega)^{-\frac{1}{2}} \quad \gamma = \frac{1}{2}\delta \ln[\omega(\omega - 1)/\hat{v}],$$
$$\lambda = \operatorname{sgn}(\hat{s}) \quad \xi = \lambda\hat{e} - \exp\left[\left(\frac{1}{2\delta} - \gamma\right)/\delta\right].$$

Then a lognormally distributed variable $X$ may be produced from a standard normally distributed variable $Z$ using the transformation

$$X = \xi + \lambda \exp\left[\frac{Z - \gamma}{\delta}\right].$$

This is simply done by replacing the second step of the procedure with

(b) Let $f_{ij} = \xi + \lambda\exp[(q_{ij} - \gamma)/\delta]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

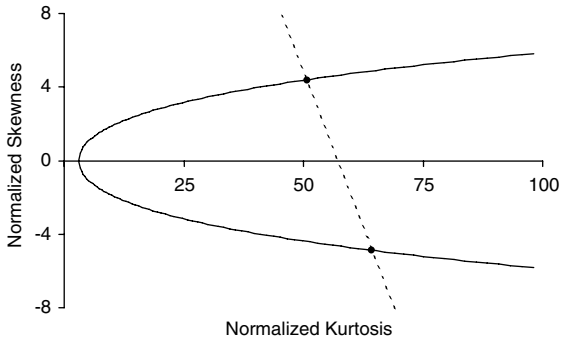


Fig. 2. Dots mark the solutions in (normalized-kurtosis, normalized-skewness) space that simultaneously satisfy conditions (20) and (21).

### 6.3. Mode $evs - \bar{e}\bar{v}$

If the required kurtosis from (20) is positive, this mode is possible by fitting a general Johnson distribution to the required moments $\hat{e}, \hat{v}, \hat{s}$ and $\hat{k}$. An iterative procedure to fit the Johnson distribution is described in Hill, Hill and Holder [15]. The drawbacks are that procedures such as in [15] are often not very efficient and may fail for numerical reasons.

## 7. Illustrations

To illustrate, 50, 200, 500 and 1000 stocks were selected at random from the S&P SuperComposite 1500. After calculating the means and standard deviations of the historical covariances and variances in each case, these values, displayed Table 1, were used as inputs to the procedure to see how well the historical distributional characteristics can be replicated in a randomly generated covariance matrix in each instance.

Running Mode $ev - \bar{e}\bar{v}$, parameters $m$, $\hat{e}$ and $\hat{v}$ for each case were easily calculated from (19), (6) and (7). Parameters $\hat{s}$, $\hat{k}$, $\omega$, $\delta$, $\gamma$, $\lambda$ and $\xi$ were computed as stipulated in Section 6. All parameter values are shown in Table 2.

Note that in the $n = 50$ case of Table 2, $m$ is greater than $n$. Whereas the number of columns of a root matrix $L$ resulting from an $LL^{\mathrm{T}}$ factorization can never exceed the number of rows, the number of columns of $F$ can easily exceed the number of rows when $n$ is small due to the structure of (19). Other than for the larger value of $m$ in the $n = 50$ case, which is mostly caused by the relatively small $v$, the values of $m$ are fairly flat across all problem sizes.

Table 1
Means and standard deviations of the historical covariances and variances that were used as inputs for the four examples of this section

| | $n$ | 50 | 200 | 500 | 1000 |
|---|---|---|---|---|---|
| Covars | $e$ | 0.00195 | 0.00245 | 0.00198 | 0.00209 |
| | $\sqrt{v}$ | 0.00203 | 0.00330 | 0.00245 | 0.00264 |
| Vars | $\bar{e}$ | 0.01521 | 0.01860 | 0.01563 | 0.01616 |
| | $\sqrt{\bar{v}}$ | 0.01227 | 0.01699 | 0.01563 | 0.01528 |

Table 2
Horizontal dimensions of $F$, first four moments of the distributions from which the $f_{ij}$ are to be drawn, parameters of the lognormal distribution from which the $f_{ij}$ are actually drawn, and total job CPU times

| $n$ | 50 | 200 | 500 | 1000 |
|---|---|---|---|---|
| $m$ | 56 | 32 | 40 | 37 |
| $\hat{e}$ | 5.893857E−3 | 8.753218E−3 | 7.034515E−3 | 7.511269E−3 |
| $\hat{v}$ | 0.239312E−3 | 0.511910E−3 | 0.341451E−3 | 0.381654E−3 |
| $\hat{s}$ | 0.014851E−3 | 0.039780E−3 | 0.026670E−3 | 0.028737E−3 |
| $\hat{k}$ | 0.002363E−3 | 0.007736E−3 | 0.005406E−3 | 0.005504E−3 |
| $\omega$ | 2.003816823 | 1.811841785 | 2.075635099 | 1.951426395 |
| $\delta$ | 1.199473886 | 1.297122818 | 1.170197346 | 1.223008890 |
| $\gamma$ | 5.419587398 | 5.164660207 | 5.140375943 | 5.191528750 |
| $\lambda$ | 1.0 | 1.0 | 1.0 | 1.0 |
| $\xi$ | −0.009546403 | −0.016357617 | −0.010782360 | −0.012517171 |
| CPU seconds | 0.040 | 0.050 | 0.170 | 0.501 |

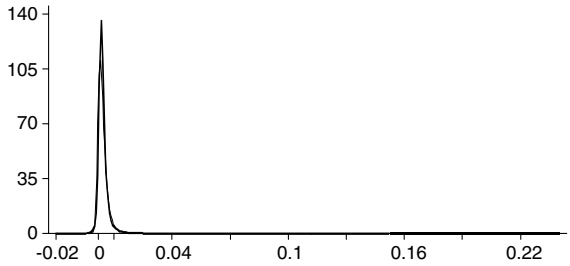


Fig. 3. Distributions of the randomly generated and historical covariances (hard to see any difference because they fall almost exactly on top of one another) for the example with $n = 1000$.

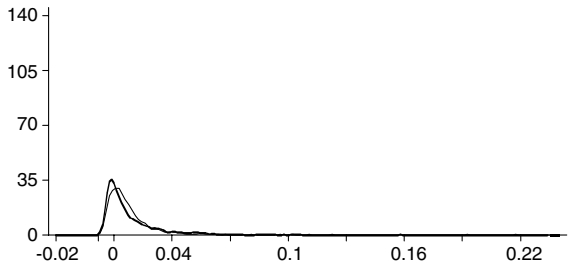


Fig. 4. Distributions of the randomly generated (thin line) and historical (thick line) variances for the example with $n = 1000$.

For the $n = 1000$ case, Fig. 3 compares the distribution of the randomly generated covariances with that of the historical covariances. The reason it is hard to observe any difference is that the two distributions fall almost perfectly on top of one another. Fig. 4 compares the distribution of the randomly generated variances (thin line) with that of the historical covariances (thick line). Here one can distinguish between the two as there is a slight difference. Figs. 3 and 4 are typical of the results one can expect to obtain. The CPU times reported in Table 2 were from running the illustrative examples on a 2.4 GHz Pentium 4 M laptop.

## 8. Situation without a procedure and concluding remarks

Before closing, we would like to comment on the difficulty one might encounter in trying to invent a valid covariance matrix without the aid of a procedure such as in this paper. To get an idea about how easy or hard this task might be, we report on the results of 20 experiments. In the experiments, the goal is to invent a valid $50 \times 50$ covariance matrix by simply randomly selecting elements from reasonable distributions.

In each trial of each experiment we populate a symmetric $50 \times 50$ matrix with off-diagonal elements randomly drawn from one reasonable distribution and diagonal elements from another. Since such matrices are positive semidefinite if and only if the determinants of all principal minors are nonnegative, we then compute the determinants of all principal minors starting at $2 \times 2$ and going to $50 \times 50$. If $(p + 1) \times (p + 1)$ is the first principal to compute negative, then we know that the $p \times p$ principal minor is the largest valid covariance matrix that can be salvaged from that trial. Utilizing values in a neighborhood about the values for $e, \sqrt{v}, \bar{e}, \sqrt{\bar{v}}$ shown in Table 1, with a sample size of 40 for each experiment, the results of the 20 experiments are shown in columns (a), (b), (c) and (d) of Table 3.

To explain, consider column (a). For the five experiments of this column, the off-diagonal elements were randomly drawn from the normal distribution with mean of 0.0016 and standard deviations of 0.0038,

Table 3
Results of 20 experiments to invent a covariance matrix

| | (a) | | (b) | | (c) | | (d) | |
|---|---|---|---|---|---|---|---|---|
| Off-diag | $N(0.0016, \sqrt{v})$ | | $N(0.0021, \sqrt{v})$ | | $N(0.0031, \sqrt{v})$ | | $N(0.0021, \sqrt{v})$ | |
| Diag | $U[0.003, 0.045]$ | | $U[0.003, 0.045]$ | | $U[0.003, 0.045]$ | | $U[0.010, 0.030]$ | |
| $\sqrt{v}$ | $p^{\text{ave}}$ | $p^{\max}$ | $p^{\text{ave}}$ | $p^{\max}$ | $p^{\text{ave}}$ | $p^{\max}$ | $p^{\text{ave}}$ | $p^{\max}$ |
| 0.0038 | 8.28 | 19 | 7.35 | 14 | 6.50 | 12 | 8.80 | 37 |
| 0.0032 | 9.18 | 17 | 8.65 | 16 | 8.55 | 20 | 10.93 | 17 |
| 0.0026 | 10.90 | 20 | 10.48 | 22 | 10.10 | 20 | 13.93 | 20 |
| 0.0020 | 14.63 | 31 | 14.15 | 26 | 11.55 | 22 | 19.65 | 27 |
| 0.0014 | 21.05 | 37 | 20.15 | 44 | 16.58 | 28 | 35.48 | 48 |

0.0032, 0.0026, 0.0020 and 0.0014. In the experiments, though, all diagonal elements were randomly drawn from the same uniform distribution with lower bound 0.003 and upper bound 0.045 as indicated. The "8.28" and "19" of the first experiment mean the following. In the 40 attempts to generate a $50 \times 50$ covariance matrix of this experiment, the largest valid covariance matrix we were able to generate was a $19 \times 19$, with an average largest valid covariance matrix size of 8.48 over the 40 attempts.

Note that over the ranges of values reasonable for the off-diagonal and diagonal elements used in the experiments, we were not successful in generating even one valid $50 \times 50$ matrix over the 800 trials of the 20 experiments. The largest we were able to generate was a $48 \times 48$ in one of the trials of the last experiment in column (d). The next largest was a $44 \times 44$ in one of the trials of the last experiment in column (b), and so forth. However, we do note that as $\sqrt{v}$ decreases, our ability to guess a covariance matrix appears to improve. But by decreasing $\sqrt{v}$ much below the lowest value of 0.0014 shown, we tend to leave the realm of what is realistic for portfolio-selection types of covariances matrices, so this does not do us much good.

Thus, if one were in need of portfolio-selection types of covariance matrices of, say, size $100 \times 100$ or greater, a capability such as described in the paper should come in handy as one's chances of being able to randomly "guess" such covariance matrices on one's own are essentially zero.

## Appendix A. Mean of variances $\geqslant$ mean of covariances

Let $\Sigma \in \mathbb{R}^{n \times n}$ be an arbitrary covariance matrix, $\mu_v$ be the mean of its $\sigma_{ii}$ diagonal (variance) elements, and $\mu_c$ be the mean of its $\sigma_{ij}, i \neq j$, off-diagonal (covariance) elements. Then $\mu_v \geqslant \mu_c$.

Knowing that in any event

$$\sigma_i^2 - 2\sigma_i\sigma_j + \sigma_j^2 = (\sigma_i - \sigma_j)^2 \geqslant 0$$

we have from Cauchy–Schwarz and the above

$$\sigma_{ij} \leqslant \sigma_i\sigma_j \leqslant \frac{\sigma_{ii} + \sigma_{jj}}{2}.$$

Now

$$\begin{aligned}\mu_c &= \frac{1}{n(n-1)} \sum_{i \neq j} \sigma_{ij} \leqslant \frac{1}{n(n-1)} \sum_{i \neq j} \frac{\sigma_{ii} + \sigma_{jj}}{2} = \frac{1}{n(n-1)} \sum_{j>i} 2\left(\frac{\sigma_{ii} + \sigma_{jj}}{2}\right) \\ &= \frac{1}{n(n-1)} \sum_{j>i} (\sigma_{ii} + \sigma_{jj}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} (n-1)\sigma_{ii} = \frac{1}{n} \sum_{i=1}^{n} \sigma_{ii} = \mu_v. \end{aligned} \tag{22}$$

Noting the inequality in (22), $\mu_v \geqslant \mu_c$.

## Appendix B. Derivation of (19) from (15)

Substituting (7) which is $\hat{v} = -\hat{e}^2 + \sqrt{\hat{e}^4 + \frac{v}{m}}$ into (15) yields

$$\bar{e} = m\sqrt{\hat{e}^4 + \frac{v}{m}}. \tag{23}$$

After substituting (6) which is $\hat{e} = \sqrt{\frac{e}{m}}$ into (23), we obtain

$$\bar{e} = m\sqrt{\frac{e^2}{m^2} + \frac{v}{m}} = \sqrt{e^2 + mv},$$

which upon algebraic manipulation yields the expression of (19)

$$m = \frac{\bar{e}^2 - e^2}{v}.$$

## References


[1] B. Aouni, F. Ben Abdelaziz, R. El-Fayedh. Chance constrained compromise programming for portfolio selection, Laboratoire LARODEC, Institut Superieur de Gestion, La Bardo 2000, Tunis, Tunisia, 2005.
[2] M. Arenas-Parra, A. Bilbao-Terol, M.V. Rodriguez-Uria, A fuzzy goal programming approach to portfolio selection, European Journal of Operational Research 133 (2) (2001) 287–297.
[3] E. Ballestero, C. Romero, Portfolio selection: A compromise programming solution, Journal of the Operational Research Society 47 (11) (1996) 1377–1386.
[4] C.A. Bana e Costa, J.O. Soares, A multicriteria model for portfolio management, European Journal of Finance 10 (3) (2004) 198–211.
[5] R.B. Bendel, M.R. Mickey, Population correlation matrices for sampling experiments, Communications in Statistics: Simulation and Computation B7 (1) (1978) 163–182.
[6] G. Bouri, J.M. Martel, H. Chabchoub, A multi-criterion approach for selecting attractive portfolio, Journal of Multi-Criteria Decision Analysis 11 (3) (2002) 269–277.
[7] P. Bratley, B. Fox, L. Schrage, A Guide to Simulation, Springer-Verlag, New York, 1987.
[8] V.K. Chopra, W.T. Ziemba, The effect of errors in means, variances, and covariances on optimal portfolio choice, Journal of Portfolio Management (Winter) (1993) 6–11.
[9] P.I. Davies, N.J. Higham, Numerically stable generation of correlation matrices and their factors, BIT 40 (4) (2000) 640–651.
[10] M. Ehrgott, K. Klamroth, C. Schwehm, An MCDM approach to portfolio optimization, European Journal of Operational Research 155 (3) (2004) 752–770.
[11] E.J. Elton, M.J. Gruber, S.J. Brown, W. Goetzmann, Modern Portfolio Theory and Investment Analysis, sixth ed., John Wiley, New York, 2002.
[12] J.B. Guerard, A. Mark, The optimization of efficient portfolios: The case for an R&D quadratic term, Research in Finance 20 (2003) 217–247.
[13] N.J. Higham, Matrix nearness problems and applications, in: M.J.C. Grover, S. Barnett (Eds.), Applications of Matrix Theory, Oxford University Press, Oxford, 1989, pp. 1–27.
[14] N.J. Higham, Computing the nearest correlation matrix—A problem for finance, IMA Journal of Numerical Analysis 22 (2002) 329–343.
[15] I.D. Hill, R. Hill, R.L. Holder, Fitting Johnson curves by moments, Applied Statistics 25 (2) (1976) 180–189.
[16] M. Hirschberger, Y. Qi, R.E. Steuer, Quadratic parametric programming for portfolio selection with random problem generation and computational experience. Department of Banking and Finance, University of Georgia, Athens, working paper.
[17] M. Hirschberger, Y. Qi, R.E. Steuer, Tri-criterion quadratic-linear programming. Department of Banking and Finance, University of Georgia, Athens, working paper.
[18] R.B. Holmes, On random correlation matrices, SIAM Journal on Matrix Analysis and Applications 12 (2) (1991) 239–272.
[19] Ch. Hurson, N. Ricci, Multicriteria decision making and portfolio management with arbitrage pricing theory, in: C. Zopounidis (Ed.), Operational Tools in The Management of Financial Risks, Kluwer Academic Publishers, Dordrecht, 1998, pp. 31–55.
[20] V. Jog, I. Kaliszewski, W. Michalowski, Using attribute trade-off information in investment, Journal of Multi-Criteria Decision Analysis 8 (4) (1999) 189–199.

[21] N.L. Johnson, Systems of frequency curves generated by methods of translation, Biometrika 36 (1949) 149–176.
[22] H. Konno, K.-I. Suzuki, A mean-variance-skewness portfolio optimization model, Journal of the Operations Research Society of Japan 38 (2) (1995) 173–187.
[23] P. Korhonen, G.-Y. Yu, A reference direction approach to multiple objective quadratic-linear programming, European Journal of Operational Research 102 (3) (1997) 601–610.
[24] L. Laloux, P. Cizeau, J.-P. Bouchaud, M. Potters, Noise dressing in financial correlation matrices, Physical Review Letters 83 (7) (1999) 1467–1470.
[25] L. Laloux, P. Cizeau, M. Potters, J.-P. Bouchaud, Random matrix theory and financial correlations, International Journal of Theoretical and Applied Finance 3 (3) (2000) 391–397.
[26] H. L'Hoir, J. Teghem, Portfolio selection by MOLP using interactive branch and bound, Foundations of Computing and Decision Sciences 20 (3) (1995) 175–185.
[27] S.P. Lin, R.B. Bendel, Algorithm AS 213: Generation of population correlation matrices with specified eigenvalues, Applied Statistics 34 (1985) 193–198.
[28] H.M. Markowitz, Portfolio selection, Journal of Finance 7 (1) (1952) 77–91.
[29] H.M. Markowitz, The optimization of a quadratic function subject to linear constraints, Naval Research Logistics Quarterly 3 (1956) 111–133.
[30] H.M. Markowitz, Portfolio Selection: Efficient Diversification in Investments, John Wiley, New York, 1959.
[31] H.M. Markowitz, G.P. Todd, Mean-Variance Analysis in Portfolio Choice and Capital Markets, Frank J. Fabozzi Associates, New Hope, Pennsylvania, 2000.
[32] G. Marsaglia, I. Olkin, Generating correlation matrices, SIAM Journal on Scientific and Statistical Computing 5 (2) (1984) 470–475.
[33] W. Ogryczak, Multiple criteria linear programming model for portfolio selection, Annals of Operations Research 97 (2000) 143–162.
[34] S. Pafka, I. Kondor, Noisy covariance matrices and portfolio optimization II, Physica (A319) (2003) 487, e-print: cond-mat/0205119.
[35] S. Pafka, M. Potters, I. Kondor, Exponential weighting and random matrix theory based filtering of financial covariance matrices for portfolio optimization. Department of Physics of Complex Systems, Eötvös University, Budapest, 2004. e-Print: cond-mat/0402573.
[36] J. Spronk, W.G. Hallerbach, Financial modelling: Where to go? with an illustration for portfolio management, European Journal of Operational Research 99 (1) (1997) 113–127.
[37] R.E. Steuer, Y. Qi, M. Hirschberger, Suitable-portfolio investors, nondominated frontier sensitivity, and the effect of multiple objectives on standard portfolio selection, Annals of Operations Research, forthcoming.
[38] M. Tamiz, R. Hasham, D.F. Jones, B. Hesni, E.K. Fargher, A two staged goal programming model for portfolio selection, in: M. Tamiz (Ed.), Lecture Notes in Economics and Mathematical Systems, vol. 432, Springer-Verlag, Berlin, 1996, pp. 386–399.
[39] L.R. Tucker, R.F. Koopman, R.F. Linn, Evaluation of factor analytic research procedures by means of simulated correlation matrices, Psychometrika 34 (1969) 421–459.
[40] C. Zopounidis, M. Doumpos, INVESTOR: A decision support system based on multiple criteria for portfolio selection and composition, in: A. Colorni, M. Paruccini, B. Roy (Eds.), A-MCD-A Aide Multi Critere a la Decision—Multiple Criteria Decision Aiding, European Commission Joint Research Centre, Brussels, 2000, pp. 371–381.