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ORIGINAL RESEARCH

An analytical derivation of properly efficient sets in multi-objective portfolio selection

Yue Qi[1] · **Ralph E. Steuer**[2]

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Abstract
Computing efficient sets has long been a topic in multiple-objective optimization and research has made substantial progress. However, there are still limitations in the multiple-objective portfolio selection and optimization areas. Firstly, researchers typically focus on models containing only one quadratic objective. Secondly, few researchers pursue multiple quadratic objectives, but their algorithms could be relatively elusive and it could be a pity that they do not explicitly demonstrate the efficient sets' structure. Lastly, researchers mostly limit their scope to three objectives. Within this context, this paper makes theoretical contributions to the literature. Operating with multiple quadratic objectives, we analytically derive closed-form formulae for the computation of the properly efficient and weakly efficient sets of problems and demonstrate the efficient sets' structure in the form of a sequence of pyramids in decision space. Although we are restricted to equality-constraint-only models, our results have implications for general-constraint models. In addition, our methods can be extended to general k-quadratic objective models.

Keywords Multi-objective portfolio selection · Multiple quadratic objectives · Properly efficient set · Weakly efficient set

JEL Classification G11 · C61

1 Introduction

Since its introduction by Markowitz (1952), the mean-variance bi-criterion model has been the dominant model behind the development of what is known today as modern portfolio

Yue Qi and Ralph E. Steuer authors contributed equally to this work.

✉ Yue Qi
yorkche@nankai.edu.cn

Ralph E. Steuer
rsteuer@uga.edu

1 Department of Financial Management, Business School, Nankai University, 94 Weijin Road, Tianjin 300071, China

2 Department of Finance, Terry College of Business, University of Georgia, Athens, GA 30602, USA

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theory MPT (see for instance the content in Elton et al. (2014)). While the bi-criterion model has had little competition throughout the completion of the 20th century, models that possess additional criteria have begun to make inroads on the bi-criterion model's domain. By this we mean that many papers have been written on the emergence of additional criteria in portfolio selection and there is the book written by Xidonas et al. (2012) which provides excellent coverage on the topic. Although the additional criteria appearing in the resulting multi-criteria models of portfolio selection were at first mostly financial, recent years have seen non-financial criteria such as social responsibility enter the picture strongly, so almost anything is possible in this regard. Thus there is a growing need for understanding as much as possible about portfolio selection models in which there are more than two criteria.

In any bi- or multi-criterion model, it is known that one's optimal solution is in the problem's set of efficient solutions which takes the form of a frontier in bi-criterion situations and a surface in multi-criterion contexts. However, early on, Geoffrion (1968) recognized that on these frontiers and surfaces there are two kinds of efficient solutions, those that are properly efficient and those that are improperly efficient. The significance of the difference between the two, from a practical point of view, is that it is highly unlikely, or in more formal terms “not rational", for an improperly efficient point to optimal. Thus it is only the set of all properly efficient solutions that investors need to be aware of in portfolio selection without having the set presented to them cluttered up with improperly efficient solutions, even though there may not be many of them.[1] This is because at an improperly efficient solution there is a tradeoff situation that would cause a rational investor to move away from the improperly efficient solution to someplace in properly efficient solution territory. This is clear in the definition of a properly efficient solution given by Geoffrion (1968). (Geoffrion (1968), p.618) concentrates on properly efficient solutions (instead of efficient solutions), because properly efficient solutions are normal (instead of anomalous) and carry satisfactory properties. Due to the technicality, we will introduce the definition in Sect. 2.

What the definition says is that if one is at an improperly efficient solution, there exist two objectives for which there is not a finite upper bound on the trade-off rate between the two. That is, initially, one could give up almost nothing in one criterion yet get almost an unlimited amount of the other criterion in relation. This is the rationale behind why it can be said that it would not be “rational" for an investor to remain at an improperly efficient solution when faced with such a prospect.

What we do in this paper, in the sense of Merton (1972), is to develop, in a mathematically tractable fashion, formulae for the computation of all properly efficient points of multi-objective portfolio selection models as essentially this is all that is needed in theory when dealing with serious investors. Of course, all efficient solutions could be accumulated at any point in time by adding all improperly efficient solutions to the set developed in the paper, so at least we always have the option to go one way or the other.

The rest of this paper is organized as follows: We review multiple-objective optimization and portfolio selection, suggest the research limitations, and present this paper's originality in Sect. 2. We justify multi-objective portfolio selection with multiple quadratic objectives in Sect. 3. We analytically derive properly efficient sets and weakly efficient sets in Sect. 4. We illustrate our derivation in Sect. 5. We extend the derivation to k-quadratic objective models in Sect. 6. We conclude this paper in Sect. 7.

[1] In bi-criterion portfolio selection improperly efficient points can only occur at the minimum variance point and the maximum return point. But in multi-criteria portfolio selection improperly efficient points can occur anywhere along the periphery of the efficient surface.

2 Background information

2.1 Multiple-objective optimization

A multiple-objective optimization problem can be modeled as

$$\begin{aligned} &\max\{z_1 = f_1(\mathbf{x})\} \\ &\quad\vdots \\ &\max\{z_k = f_k(\mathbf{x})\} \\ &\text{s.t. } \mathbf{x} \in S \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a decision vector in decision space, k is the number of objectives, $f_1(\mathbf{x}) \ldots f_k(\mathbf{x})$ are objective functions, $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix}$ is a criterion vector in criterion space (i.e., in the space of the objectives), $S \subset \mathbb{R}^n$ is the feasible region in decision space, and $Z = \{\mathbf{z} \mid z_1 = f_1(\mathbf{x}) \ldots z_k = f_k(\mathbf{x}),\ \mathbf{x} \in S\}$ is the feasible region in criterion space. Then, we introduce the following definitions:

Definition 1 For $\bar{\mathbf{z}} \in Z$ and $\mathbf{z} \in Z$, that $\bar{\mathbf{z}}$ *dominates* $\mathbf{z}$ is defined as $\bar{z}_1 \geq z_1 \ldots \bar{z}_k \geq z_k$ with at least one inequality strict.

Definition 2 That $\bar{\mathbf{z}} \in Z$ is *nondominated* is defined as that there does not exist a $\mathbf{z} \in Z$ such that $\mathbf{z}$ dominates $\bar{\mathbf{z}}$. Then if $\bar{\mathbf{x}} \in S$ is an inverse image of $\bar{\mathbf{z}}$ (i.e., $\bar{\mathbf{z}} = \begin{bmatrix} f_1(\bar{\mathbf{x}}) \\ \vdots \\ f_k(\bar{\mathbf{x}}) \end{bmatrix}$), $\bar{\mathbf{x}}$ is *efficient*.

Definition 3 That $\bar{\mathbf{x}} \in S$ is *properly efficient* is defined as that $\bar{\mathbf{x}}$ is efficient and there exists a scalar $M > 0$ such that for each $i \in \{1, \ldots, k\}$, $\frac{f_i(\mathbf{x}) - f_i(\bar{\mathbf{x}})}{f_j(\bar{\mathbf{x}}) - f_j(\mathbf{x})} \leq M$ for some $j \in \{1, \ldots, k\}$ such that $f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}})$ whenever $\mathbf{x} \in S$ and $f_i(\mathbf{x}) > f_i(\bar{\mathbf{x}})$. Then if $\bar{\mathbf{z}}$ is the criterion vector of $\bar{\mathbf{x}}$, $\bar{\mathbf{z}}$ is *properly nondominated*.

Definition 4 That $\bar{\mathbf{z}} \in Z$ is *weakly nondominated* is defined as that there does not exist a $\mathbf{z} \in Z$ such that $z_1 > \bar{z}_1 \ldots z_k > \bar{z}_k$. The inverse image of $\bar{\mathbf{z}}$ is $\bar{\mathbf{x}} \in S$. $\bar{\mathbf{x}}$ is *weakly efficient*.

The sets of all efficient points, properly efficient points, and weakly efficient points are respectively called the efficient set, properly efficient set, and weakly efficient set. The sets of all nondominated points, properly nondominated points, and weakly nondominated points are respectively called the nondominated set, properly nondominated set, and weakly nondominated set. The purpose of multiple-objective optimization is to compute the efficient set, properly efficient set, and weakly efficient set for the discretion of decision makers. Then, a final solution in the efficient set, properly efficient set, and weakly efficient set can be pinpointed in accordance with the decision maker's preferences.

To solve (1), mechanisms to convert (1) to an ordinary single-objective program are utilized. One common mechanism is a weighted-sums method. By this method, based on a weighting vector $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} \in \mathbb{R}^k$, we form the weighted-sums program in (2).

$$\max\{z_w = \lambda_1 f_1(\mathbf{x}) + \ldots + \lambda_k f_k(\mathbf{x})\}$$

$$\text{s.t. } \mathbf{x} \in S. \tag{2}$$

For properly efficient sets, (Geoffrion (1968), pp.619–621) proves the relationship between (2) and (1) in the following theorem:

Theorem 1 *For (1), let S be a convex set and let all of $f_1(\mathbf{x}) \ldots f_k(\mathbf{x})$ be concave functions. Then, $\bar{\mathbf{x}} \in S$ is properly efficient in (1) if and only if there exists a $\bar{\boldsymbol{\lambda}} > \mathbf{0}$ such that $\bar{\mathbf{x}}$ is the optimal solution of (2) with the $\bar{\boldsymbol{\lambda}}$.* [2]

For weakly efficient sets, (Ehrgott (2005), p.71) proves the relationship between (2) and (1) in the following theorem:

Theorem 2 *If $\bar{\mathbf{x}}$ is the optimal solution of (2) for a $\bar{\boldsymbol{\lambda}} \gneq \mathbf{0}$, $\bar{\mathbf{x}} \in S$ is weakly efficient in (1).*[3] *Suppose that for (1), S is a convex set and $f_1(\mathbf{x}) \ldots f_k(\mathbf{x})$ are concave functions. If $\bar{\mathbf{x}} \in S$ is weakly efficient in (1), there exists a $\bar{\boldsymbol{\lambda}} \gneq \mathbf{0}$ such that $\bar{\mathbf{x}}$ is the optimal solution of (2) with the $\bar{\boldsymbol{\lambda}}$.*

Another common mechanism is the e-constraint method. By this approach, only one objective is retained while the others are transformed into constraints as follows:

$$\begin{aligned} &\max \{z_1 = f_1(\mathbf{x})\} \\ &\text{s.t. } f_2(\mathbf{x}) = e_2 \\ &\qquad \vdots \\ &\qquad f_k(\mathbf{x}) = e_k \\ &\qquad \mathbf{x} \in S \end{aligned}$$

where $e_2 \ldots e_k$ are right-hand-side parameters.

2.2 Portfolio selection and the justification

Let us now consider the above in a portfolio context. For n assets, let $\mathbf{r}_1 = \begin{bmatrix} r_{11} \\ \vdots \\ r_{1n} \end{bmatrix}$ be an n-vector of individual asset returns. For a portfolio weight vector $\mathbf{x} \in S$, the portfolio return is r_1 as follows:

$$r_1 = \mathbf{r}_1^T \mathbf{x} \tag{3}$$

Markowitz (1952) formulates portfolio selection as

$$\begin{aligned} &\min \{z_1 = var(r_1) = \mathbf{x}^T \boldsymbol{\Sigma}_1 \mathbf{x}\} \\ &\max \{z_2 = E(r_1) = \boldsymbol{\mu}_1^T \mathbf{x}\} \\ &\text{s.t. } \mathbf{x} \in S \end{aligned} \tag{4}$$

where $var()$ is variance, $E()$ is expectation, $\boldsymbol{\Sigma}_1$ is an $n \times n$ covariance matrix of asset returns, and $\boldsymbol{\mu}_1 = \begin{bmatrix} \mu_{11} \\ \vdots \\ \mu_{1n} \end{bmatrix}$ is a vector of asset expected returns.

[2] For $\bar{\boldsymbol{\lambda}} = \left[\bar{\lambda}_1 \ \ldots \ \bar{\lambda}_k\right]^T \in \mathbb{R}^k$ and $\mathbf{0} = \left[0 \ \ldots \ 0\right]^T \in \mathbb{R}^k$, $\bar{\boldsymbol{\lambda}} > \mathbf{0}$ denotes that $\bar{\lambda}_1 > 0 \ldots \bar{\lambda}_k > 0$.

[3] For $\bar{\boldsymbol{\lambda}} = \left[\bar{\lambda}_1 \ \ldots \ \bar{\lambda}_k\right]^T \in \mathbb{R}^k$ and $\mathbf{0} = \left[0 \ \ldots \ 0\right]^T \in \mathbb{R}^k$, $\bar{\boldsymbol{\lambda}} \gneq \mathbf{0}$ denotes that $\bar{\lambda}_1 \geq 0 \ldots \bar{\lambda}_k \geq 0$ with $\lambda_i \neq 0$ for some $i \in \{1, ..., k\}$.

Traditional financial economists (e.g., (Huang and Litzenberger (1988), pp.59–60)) formulate portfolio selection as

$$\begin{aligned} &\max\{E(u(r_1))\} \\ &\text{s.t. } \mathbf{x} \in S \end{aligned} \tag{5}$$

where $u(r_1)$ is a utility function (i.e., defined as increasing and concave w.r.t. r_1). (Huang and Litzenberger (1988), p.61) justify (4) by (5) in the following theorem:

Theorem 3 *For a quadratic utility function $u(r_1) = r_1 - \frac{1}{2}qr_1^2$ with $q > 0$ as a parameter such that $r_1 \leq \frac{1}{q}$ in order to for $u(r_1)$ over the domain of the problem, the optimal solution of (5) is an efficient solution of (4).*

Ben Abdelaziz and La Torre (2023) contemplate generalized utility for continuous-time portfolio selection. Merton (1972) and Roll (1977) analyze the following model:

$$\begin{aligned} &\min\{z_1 = \mathbf{x}^T \mathbf{\Sigma}_1 \mathbf{x}\} \\ &\max\{z_2 = \boldsymbol{\mu}_1^T \mathbf{x}\} \\ &\text{s.t. } \mathbf{1}^T \mathbf{x} = 1 \end{aligned} \tag{6}$$

where $\mathbf{1}$ is an $n \times 1$ vector of ones. Merton (1972) makes the following assumption:

Assumption 1 $\mathbf{\Sigma}_1$ is invertible and thus positive definite.

Merton (1972) analytically derives the efficient set by applying an e-constraint method. The analyticity brings substantial advantage in research and teaching (as appraised by (Huang and Litzenberger (1988), pp.60–62)), and (6) serves as the foundation of capital asset pricing models. Assume $\mathbf{\Sigma}_1$ as invertible and thus positive definite. [4] The relevant formulae are as follows: The *minimum-variance portfolio* is

$$\mathbf{x}^0 = \frac{1}{f}\mathbf{\Sigma}_1^{-1}\mathbf{1}. \tag{7}$$

where $f = \mathbf{1}^T \mathbf{\Sigma}_1^{-1}\mathbf{1}$ and $c = \boldsymbol{\mu}_1^T \mathbf{\Sigma}_1^{-1}\mathbf{1}$. The efficient set is

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \lambda_2(\mathbf{\Sigma}_1^{-1}\boldsymbol{\mu}_1 - \frac{c}{f}\mathbf{\Sigma}_1^{-1}\mathbf{1}) + \mathbf{x}^0, \lambda_2 \in [0, \infty)\}. \tag{8}$$

With $\lambda_2 \in [0, \infty)$, (8) is a 1-dimensional pyramid (explained later).

2.3 Multiple-objective portfolio selection and optimization

Markowitz (1991), pp.471&476 notices extra objectives in addition to z_1 and z_2 of portfolio selection (4). Sharpe (2001) also notices the objectives and incorporates them into utility functions. Fama (1996), pp.445–447 and Cochrane (2011), pp.1081–1082 focus on multiple factors for asset pricing and further propose the factors' risks as objectives.

Fama (1996), Steuer et al. (2007), Dorfleitner et al. (2012), Hirschberger et al. (2013), Utz et al. (2015), Qi et al. (2017), and Qi and Steuer (2020) extend (4) and regulate the extra objectives in multiple-objective portfolio selection as follows:

$$\min\{z_1 = var(r_1) = \mathbf{x}^T \mathbf{\Sigma}_1 \mathbf{x}\}$$

[4] A positive semidefinite matrix is a covariance matrix and vice versa as documented by Brockwell and Davis (1991).

$$
\begin{aligned}
&\max \{z_2 = E(r_1) = \boldsymbol{\mu}_1^T \mathbf{x}\} \\
&\max \{z_3 = E(r_2) = \boldsymbol{\mu}_2^T \mathbf{x}\} \\
&\qquad \vdots \\
&\max \{z_k = E(r_{k-1}) = \boldsymbol{\mu}_{k-1}^T \mathbf{x}\} \\
&\text{s.t.}\, \mathbf{x} \in S
\end{aligned} \tag{9}
$$

where $\mathbf{r}_2 \ldots \mathbf{r}_{k-1}$ are n-vectors of individual general stock objectives (e.g., ESG). For a portfolio weight vector $\mathbf{x} \in S$, the portfolio objectives are $r_2 \ldots r_{k-1}$ as extensions of (3) as follows:

$$
r_2 = \mathbf{r}_2^T \mathbf{x} \quad \ldots \quad r_{k-1} = \mathbf{r}_{k-1}^T \mathbf{x} \tag{10}
$$

$\boldsymbol{\mu}_2 \ldots \boldsymbol{\mu}_{k-1}$ are vectors of general stock expected objectives. $z_3 \ldots z_k$ measure the general portfolio expected objectives.

Salas-Molina et al. (2018) harness cash management under multiple objectives. Overall, Steuer and Na (2003), Zopounidis et al. (2015), Masmoudi and Ben Abdelaziz (2018), and Aouni et al. (2018) offer surveys.

More broadly, Miettinen (1999) and Ehrgott (2005) investigate nonlinear multiple-objective optimization. Bank et al. (1983) and Pistikopoulos et al. (2021) explore parametric-quadratic programming and open the avenue for multiple-objective portfolio selection with quadratic objectives (instead of only one quadratic objective in (9)). Particularly, Goh and Yang (1996), Jayasekara et al. (2019), and Jayasekara et al. (2023) dissect such multiple quadratic objectives.

2.4 Research limitations of multiple-objective portfolio optimization

The researchers above substantially improve multiple-objective portfolio selection and optimization. However, there could be the following research limitations:

Firstly, researchers (e.g., Hirschberger et al. (2013) and Utz et al. (2015)) typically exploit models with only one quadratic objective.

Secondly, few researchers pursue multiple quadratic objectives, but their algorithms could be relatively elusive and it could be a pity that they do not explicitly demonstrate the efficient sets' structure. For instance, Goh and Yang (1996) resolve multiple quadratic objectives by launching active-set algorithms. Jayasekara et al. (2019) review the scalarization methods and instigate modified hybrid methods. Jayasekara et al. (2023) further push the boundary by generalized scalarization and computational study. However, the three teams do not explicitly demonstrate the efficient sets' structure (e.g., the sets' composition).

Lastly, researchers typically limit the scope to three objectives. For example, Hirschberger et al. (2013) initiate their algorithms for three objectives only.

2.5 Originality of this paper

With regard to the literature, this paper falls into the analytical derivation stream of portfolio selection as given its start in Merton (1972) and Roll (1977). There is a related but different stream in the continuous-time world as addressed in the recent paper by Ben Abdelaziz and La Torre (2023) but that is not covered here. As Markowitz solution approaches are cumbersome because of their inequality constraints, the purpose of the analytical derivation stream relevant to this paper, interchangeably called *efficient set mathematics*, is to enable the computation of portfolio solutions in much less time. This is accomplished by only allowing equality constraints thereby facilitating the development of closed-form formulas for various quantities in mean-variance portfolio selection. For instance, out of these formulas we have (7), (8) for the minimum-variance portfolio and efficient set, respectively.

Given that Merton (1972) and Roll (1977) had done such a complete job on this body of knowledge, other than for a few additional contributions such as by Best and Grauer (1990), little has been added to the stream since, the situation being that once a theorem is proved, it is proved for eternity. While one might feel that this would cause academic interest in the area to wane, this has not been the case. This is because it has been found that the best way to teach theory in portfolio selection is through the use of efficient set mathematics as reflected in textbooks such as by Huang and Litzenberger (1988), Campbell et al. (1997), Cochrane (2005), and Back (2017), books of the type that are heavily studied in all graduate programs in finance around the world.

However, recent years have seen substantial changes in portfolio selection with the surge in interest in additional criterion concerns (beyond risk and return) such as sustainability, renewable energy, and even Shariah compliance in Islamic investing Masri (2018). Whereas, looking to the past, the classical model of portfolio selection is the bi-criterion model of risk and return, looking to the future, it may well be that the classical model of portfolio selection should be at least a tri-criterion one to be flexible enough to accommodate the increasing numbers of investors that wish to have additional criteria incorporated into their investing.

All of this has opened the door for it to be recognized that problems of portfolio selection can easily have more objectives than just traditional risk and return going forward giving rise to the need for new activity in the analytical derivation stream started by Merton (1972) and Roll (1977) for portfolio problems with more than usual risk and return as objectives. While there has already been some work done in the area in the form of Qi et al. (2017) and Qi and Steuer (2020), results are not as easily obtained as in Merton (1972) and Roll (1977) as the efficient set becomes a surface when the number of criteria becomes more than two. Thus, the contribution of this paper is that it presents new closed-form results in support of the emerging three-or-more objective portfolio selection situations that give evidence of appearing at a rapid rate.

3 Justifying multiple-objective portfolio selection with quadratic objectives

In this section, we justify multiple-objective portfolio selection with quadratic objectives by extending Theorem 3. We extend (9) by appending a general quadratic objective as follows:

$$\begin{aligned} &\min \{z_1 = var(r_1) = \mathbf{x}^T \boldsymbol{\Sigma}_1 \mathbf{x}\} \\ &\min \{z_2 = var(r_2) = \mathbf{x}^T \boldsymbol{\Sigma}_2 \mathbf{x}\} \\ &\max \{z_3 = E(r_1) = \boldsymbol{\mu}_1{}^T \mathbf{x}\} \\ &\max \{z_4 = E(r_2) = \boldsymbol{\mu}_2{}^T \mathbf{x}\} \\ &\quad \text{s.t. } \mathbf{x} \in S \end{aligned} \tag{11}$$

where r_1 is introduced in (3). r_2 is introduced in (10). $\boldsymbol{\Sigma}_2$ is a covariance matrix of r_2. $\boldsymbol{\mu}_2$ is introduced in (9).

A method to justify (11) is to formulate r_1 and r_2 by the following stochastic multiple-objective programming model

$$\begin{aligned} &\max\{r_1\} \\ &\max\{r_2\} \\ &\text{s.t.}\, \mathbf{x} \in S. \end{aligned} \tag{12}$$

By the research of Caballero et al. (2001), a variance-expectation operationalization of (12) is (11).

Another method to justify (11) is to formulate r_1 and r_2 by extended utility function. Suppose that the utility function of r_1 is $u_1(r_1) = r_1 - \frac{1}{2}q_1r_1^2$ with $q_1 > 0$, the utility function of r_2 is $u_2(r_2) = r_2 - \frac{1}{2}q_2r_2^2$ with $q_2 > 0$, and the extended utility function of (r_1, r_2) is as follows:

$$u(r_1, r_2) = u_1(r_1) + u_2(r_2) = r_1 - \frac{1}{2}q_1r_1^2 + r_2 - \frac{1}{2}q_2r_2^2. \tag{13}$$

We demonstrate that the joint extended utility function inherits the increasing and concave properties of the individual utility functions in the following lemma:

Lemma 1 *For fixed r_2, $u(r_1, r_2)$ is increasing w.r.t. r_1 if $r_1 \leq \frac{1}{q_1}$. For fixed r_1, $u(r_1, r_2)$ is increasing w.r.t. r_2 if $r_2 \leq \frac{1}{q_2}$. Moreover, $u(r_1, r_2)$ is concave w.r.t. (r_1, r_2).*

Proof The increasing property is inherited from that of $u_1(r_1)$ and $u_2(r_2)$. We can prove the concavity of $u(r_1, r_2)$ by directly using the definition of concavity and utilizing $u_1(r_1)$'s and $u_2(r_2)$'s concavity. □

Portfolio selection can be formulated by the extended utility function as

$$\begin{aligned} &\max\{E(u(r_1, r_2))\} \\ &\text{s.t.}\, \mathbf{x} \in S. \end{aligned} \tag{14}$$

Then, we extend Theorem 3 and justify (11) by (14) in the following theorem:

Theorem 4 *For $u(r_1, r_2)$, if $r_1 \leq \frac{1}{q_1}$ and $r_2 \leq \frac{1}{q_2}$, the optimal solution of (14) is an efficient solution of (11).*

Proof We compute $E(u(r_1, r_2))$ by (13) as follows:

$$\begin{aligned} E(u(r_1, r_2)) &= E(r_1) - \frac{1}{2}q_1E(r_1^2) + E(r_2) - \frac{1}{2}q_2E(r_2^2) \\ &= E(r_1) - \frac{1}{2}q_1(E(r_1))^2 - \frac{1}{2}q_1 var(r_1) + E(r_2) - \frac{1}{2}q_2(E(r_2))^2 - \frac{1}{2}q_2 var(r_2) \end{aligned}$$

Because $r_1 \leq \frac{1}{q_1}$ and $r_2 \leq \frac{1}{q_2}$, $E(r_1) \leq \frac{1}{q_1}$ and $E(r_2) \leq \frac{1}{q_2}$. Then, $E(u(r_1, r_2))$ is strictly increasing w.r.t. only $E(r_1)$ with $E(r_1) \leq \frac{1}{q_1}$, and $E(u(r_1, r_2))$ is strictly increasing w.r.t. only $E(r_2)$ with $E(r_2) \leq \frac{1}{q_2}$. $E(u(r_1, r_2))$ is strictly decreasing w.r.t. only $var(r_1)$, and $E(u(r_1, r_2))$ is strictly decreasing w.r.t. only $var(r_2)$.

With $\bar{\mathbf{x}}$ the optimal solution of (14), its criterion vector in (11) is $\begin{bmatrix} var(\bar{r}_1) \\ var(\bar{r}_2) \\ E(\bar{r}_1) \\ E(\bar{r}_2) \end{bmatrix}$. Suppose that $\bar{\mathbf{x}}$ is not an efficient solution of (11); i.e., there exists an $\mathbf{x}' \in S$ with the criterion vector of

(11) as $\begin{bmatrix} var(r'_1) \\ var(r'_2) \\ E(r'_1) \\ E(r'_2) \end{bmatrix}$ such that $\begin{bmatrix} var(r'_1) \\ var(r'_2) \\ E(r'_1) \\ E(r'_2) \end{bmatrix}$ dominates $\begin{bmatrix} var(\bar{r}_1) \\ var(\bar{r}_2) \\ E(\bar{r}_1) \\ E(\bar{r}_2) \end{bmatrix}$. Because of the dominance relationship, two strictly increasing relationships above, and two strictly decreasing relationships above, $E(u(r'_1, r'_2)) > E(u(\bar{r}_1, \bar{r}_2))$. However, this "$>$" contradicts the fact that $\bar{\mathbf{x}}$ is the optimal solution of (14). The supposition is incorrect. Therefore, $\bar{\mathbf{x}}$ is an efficient solution of (11). □

4 Analytically deriving properly efficient sets and weakly efficient sets

We further extend (11) as follows:

$$\begin{aligned} \min \{z_1 &= \mathbf{x}^T \boldsymbol{\Sigma}_1 \mathbf{x}\} \\ \min \{z_2 &= \mathbf{x}^T \boldsymbol{\Sigma}_2 \mathbf{x}\} \\ \max \{z_3 &= \mathbf{c}_1{}^T \mathbf{x}\} \\ \max \{z_4 &= \mathbf{c}_2{}^T \mathbf{x}\} \\ &\vdots \\ \max \{z_{2+l} &= \mathbf{c}_l{}^T \mathbf{x}\} \\ \text{s.t. } \mathbf{A}^T \mathbf{x} &= \mathbf{b} \end{aligned} \tag{15}$$

In (11), variances (e.g., $var(r_1)$) is attached to expectations ($E(r_1)$). Steuer et al. (2007) and Dorfleitner et al. (2012) recommend some form of the detachment. Therefore, investors carry the following choices in (15):

- Investors can still attach variances to expectations by setting $\mathbf{c}_1 = \boldsymbol{\mu}_1$ and $\mathbf{c}_2 = \boldsymbol{\mu}_2$.
- Otherwise, investors can moderately detach variances and expectations by setting $\mathbf{c}_1 = \boldsymbol{\mu}_1$ but $\mathbf{c}_2$ as general.

$\mathbf{c}_3 \ldots \mathbf{c}_l$ can be the vectors of asset expected values of other criteria outlined, for example, in Steuer et al. (2007).

(15) contains more objectives than (11). (15) extends (6) by including more quadratic objectives and linear objectives and by generalizing $\mathbf{1}^T \mathbf{x} = 1$ to $\mathbf{A}^T \mathbf{x} = \mathbf{b}$ with $\mathbf{A}$ as being $n \times m$. Denote $\mathbf{C} = \left[\mathbf{c}^1 \ldots \mathbf{c}^l\right]$ as an $n \times l$ matrix. Then, we rewrite (15) as

$$\begin{aligned} \min \{z_1 &= \mathbf{x}^T \boldsymbol{\Sigma}_1 \mathbf{x}\} \\ \min \{z_2 &= \mathbf{x}^T \boldsymbol{\Sigma}_2 \mathbf{x}\} \\ \max \left\{ \begin{bmatrix} z_3 \\ \vdots \\ z_{2+l} \end{bmatrix} \right. &= \left. \mathbf{C}^T \mathbf{x} \right\} \\ \text{s.t. } \mathbf{A}^T \mathbf{x} &= \mathbf{b}. \end{aligned}$$

Matrix $\left[\mathbf{A} \middle| \mathbf{C}\right]$ is $n \times (m + l)$.[5] We extend Assumption 1 and the assumption of (Qi and Steuer (2020), p.525) as follows:

Assumption 2 $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are invertible and thus positive definite.

[5] In $\left[\mathbf{A} \middle| \mathbf{C}\right]$, symbol | denotes matrix partitions.

Assumption 3 Matrix $\begin{bmatrix} \mathbf{A} | \mathbf{C} \end{bmatrix}$ has full column rank $m + l$ (i.e., $rank(\begin{bmatrix} \mathbf{A} | \mathbf{C} \end{bmatrix}) = m + l$).

By Assumption 3 and $\begin{bmatrix} \mathbf{A} | \mathbf{C} \end{bmatrix}$ as an $n \times (m + l)$ matrix, we know $n \geq (m + l)$. Moreover, $\mathbf{A}$ also has full column rank (i.e., $rank(\mathbf{A}) = m$), so the system of linear equations $\mathbf{A}^T \mathbf{x} = \mathbf{b}$ is always solvable (i.e., S is not empty).

We apply a weighted-sums method to (15) and get

$$\begin{aligned} &\min \{\lambda_1 \mathbf{x}^T \mathbf{\Sigma}_1 \mathbf{x} + \lambda_2 \mathbf{x}^T \mathbf{\Sigma}_2 \mathbf{x} - \boldsymbol{\lambda}_3^T \mathbf{C}^T \mathbf{x}\} \\ &\text{s.t. } \mathbf{A}^T \mathbf{x} = \mathbf{b} \end{aligned} \tag{16}$$

where $\boldsymbol{\lambda}_3 = \begin{bmatrix} \lambda_3 \\ \vdots \\ \lambda_{2+l} \end{bmatrix}$, and $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix}$ is a weighting vector for the objectives of (15). We respectively denote the properly efficient set and weakly efficient set of (15) as $PE(15)$ and $WE(15)$. Then, we prove that the properly efficient set of (15) and weakly efficient set of (15) can be respectively obtained by all the optimal solutions of (16) with all $\boldsymbol{\lambda} > \mathbf{0}$ (where $\mathbf{0}$ is a $(2 + l) \times 1$ vector of zeros) and by all the optimal solutions of (16) with all $\boldsymbol{\lambda} \gneq \mathbf{0}$ in the following theorem:

Theorem 5 *$PE(15)$={all the optimal solutions of (16) with all $\boldsymbol{\lambda} > \mathbf{0}$}.*
$WE(15)$={all the optimal solutions of (16) with all $\boldsymbol{\lambda} \gneq \mathbf{0}$}.

Proof For (15), $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}^T \mathbf{x} = \mathbf{b}\}$ is a convex set, and all of $f_1(\mathbf{x}) \ldots f_k(\mathbf{x})$ are convex functions (we use min in (16)). For any $\bar{\mathbf{x}} \in PE(15)$, there exists a $\bar{\boldsymbol{\lambda}} > \mathbf{0}$ such that $\bar{\mathbf{x}}$ is the optimal solution of (2) with the $\bar{\boldsymbol{\lambda}}$ by Theorem 1, so $\bar{\mathbf{x}} \in$ {all optimal solutions of (16) with $\boldsymbol{\lambda} > \mathbf{0}$} and $PE(15) \subseteq$ {all optimal solutions of (16) with $\boldsymbol{\lambda} > \mathbf{0}$}.
For any $\bar{\mathbf{x}} \in$ {all optimal solutions of (16) with $\boldsymbol{\lambda} > \mathbf{0}$}, there exists a $\bar{\boldsymbol{\lambda}} > \mathbf{0}$ such that $\bar{\mathbf{x}}$ is the optimal solution of (2) with the $\bar{\boldsymbol{\lambda}}$, so $\bar{\mathbf{x}} \in S$ is properly efficient in (1) by Theorem 1.
Therefore, $\bar{\mathbf{x}} \in PE(15)$ and {all optimal solutions of (16) with $\boldsymbol{\lambda} > \mathbf{0}$} $\subseteq PE(15)$.
Therefore, $PE(15)$={all optimal solutions of (16) with $\boldsymbol{\lambda} > \mathbf{0}$}. Similarly, $WE(15)$={all optimal solutions of (16) with $\boldsymbol{\lambda} \gneq \mathbf{0}$} by Theorem 2. ◻

For $\begin{bmatrix} \lambda_1' \\ \lambda_2' \\ \boldsymbol{\lambda}_3' \end{bmatrix} > \mathbf{0}$, we can always divide the vector by λ_1', get $\begin{bmatrix} 1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix}$, and simplify (16) as

$$\begin{aligned} &\min \{\mathbf{x}^T \mathbf{\Sigma}_1 \mathbf{x} + \lambda_2 \mathbf{x}^T \mathbf{\Sigma}_2 \mathbf{x} - \boldsymbol{\lambda}_3^T \mathbf{C}^T \mathbf{x}\} \\ &\text{s.t. } \mathbf{A}^T \mathbf{x} = \mathbf{b}. \end{aligned} \tag{17}$$

To solve (17), we apply the method of Lagrange multipliers and construct

$$L(\mathbf{x}, \mathbf{l}) = \mathbf{x}^T (\mathbf{\Sigma}_1 + \lambda_2 \mathbf{\Sigma}_2) \mathbf{x} - \boldsymbol{\lambda}_3^T \mathbf{C}^T \mathbf{x} + \mathbf{l}^T (\mathbf{A}^T \mathbf{x} - \mathbf{b}) \tag{18}$$

where $\mathbf{l} = \begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix}$ is an $m \times 1$ Lagrange-multiplier vector. Let

$$\mathbf{\Delta} = \mathbf{\Sigma}_1 + \lambda_2 \mathbf{\Sigma}_2. \tag{19}$$

By Assumption 2 and $\lambda_2 \geq 0$, $\mathbf{\Delta}$ is positive definite. Because $\mathbf{x}^T \mathbf{\Sigma} \mathbf{x}$ is positive semidefinite w.r.t. $\mathbf{x}$, $\mathbf{x}$ is the optimal solution of (18) if and only if

$$\frac{\partial L}{\partial \mathbf{x}} = 2\mathbf{\Delta}\mathbf{x} - \mathbf{C}\boldsymbol{\lambda}_3 + \mathbf{A}\mathbf{l} = \mathbf{0} \tag{20}$$

and

$$\frac{\partial L}{\partial \mathbf{l}} = \mathbf{A}^T\mathbf{x} - \mathbf{b} = \mathbf{0}. \tag{21}$$

We premultiply (20) by $\boldsymbol{\Delta}^{-1}$, get $\mathbf{x} = \frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{C}\boldsymbol{\lambda}_3 - \mathbf{A}\mathbf{l})$, substitute $\mathbf{x}$ into (21), and obtain

$$\begin{aligned} \mathbf{A}^T\frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{C}\boldsymbol{\lambda}_3 - \mathbf{A}\mathbf{l}) - \mathbf{b} = \mathbf{0} \\ \mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{C}\boldsymbol{\lambda}_3 - (\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})\mathbf{l} - 2\mathbf{b} = \mathbf{0} \\ (\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})\mathbf{l} = \mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{C}\boldsymbol{\lambda}_3 - 2\mathbf{b}. \end{aligned} \tag{22}$$

Matrix $\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A}$ is $m \times m$. We prove it as a covariance matrix in the following lemma:

Lemma 2 $\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A}$ *is a covariance matrix and moreover positive definite and thus invertible.*

Proof With $\boldsymbol{\Delta}$ positive definite, so is $\boldsymbol{\Delta}^{-1}$ as documented by Lax (2007), $\boldsymbol{\Delta}^{-1}$ can function as a covariance matrix as documented by Brockwell and Davis (1991); i.e., there exists a random vector $\mathbf{v} \in \mathbb{R}^n$ such that the covariance matrix of $\mathbf{v}$ is $\boldsymbol{\Delta}^{-1}$. The covariance matrix of $\mathbf{A}^T\mathbf{v}$ is $\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A}$.

For any $\mathbf{y} \neq \mathbf{0} \in \mathbb{R}^m$, $\mathbf{y}^T(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})\mathbf{y} = (\mathbf{A}\mathbf{y})^T\boldsymbol{\Delta}^{-1}(\mathbf{A}\mathbf{y})$. Let $\mathbf{u} = \mathbf{A}\mathbf{y}$. We know $\mathbf{u} \neq \mathbf{0}$ because $\mathbf{A}$ has full column rank by Assumption 3. Then, $\mathbf{y}^T(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})\mathbf{y} = \mathbf{u}^T\boldsymbol{\Delta}^{-1}\mathbf{u} > 0$ because $\boldsymbol{\Delta}^{-1}$ is positive definite. Therefore $\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A}$ is positive definite and thus invertible. □

We premultiply (22) by $(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}$, get $\mathbf{l} = (\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{C}\boldsymbol{\lambda}_3 - 2\mathbf{b})$, substitute $\mathbf{l}$ into $\mathbf{x} = \frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{C}\boldsymbol{\lambda}_3 - \mathbf{A}\mathbf{l})$, and obtain the optimal solution of (16) as

$$\begin{aligned} \mathbf{x} &= \frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{C}\boldsymbol{\lambda}_3 - \mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{C}\boldsymbol{\lambda}_3 - 2\mathbf{b})) \\ &= \frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{C}\boldsymbol{\lambda}_3 - \mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{C}\boldsymbol{\lambda}_3 + 2\mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{b}) \\ &= \frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{I}_n\mathbf{C}\boldsymbol{\lambda}_3 - \mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{C}\boldsymbol{\lambda}_3) + \boldsymbol{\Delta}^{-1}\mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{b} \\ &= \frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{I}_n - \mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\boldsymbol{\Delta}^{-1})\mathbf{C}\boldsymbol{\lambda}_3 + \boldsymbol{\Delta}^{-1}\mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{b} \end{aligned} \tag{23}$$

where $\mathbf{I}_n$ is an $n \times n$ identity matrix. $\boldsymbol{\Delta}^{-1}(\mathbf{I}_n - \mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\boldsymbol{\Delta}^{-1})\mathbf{C}$ is an $n \times l$ matrix. We denote the latter matrix's columns as

$$\frac{1}{2}\boldsymbol{\Delta}^{-1}(\mathbf{I}_n - \mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\boldsymbol{\Delta}^{-1})\mathbf{C} = \left[\mathbf{h}_3 \ldots \mathbf{h}_{2+l}\right] \tag{24}$$

where $\mathbf{h}_3 \ldots \mathbf{h}_{2+l}$ are $n \times 1$ vectors. Because $\boldsymbol{\Delta} = \boldsymbol{\Sigma}_1 + \lambda_2\boldsymbol{\Sigma}_2$, $\mathbf{h}_3 \ldots \mathbf{h}_{2+l}$ are functions of λ_2.

Then by (23) and $\boldsymbol{\lambda}_3 = \begin{bmatrix} \lambda_3 \\ \vdots \\ \lambda_{2+l} \end{bmatrix}$, the properly efficient set of (15) is

$$\begin{aligned} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \lambda_3\mathbf{h}_3 + \ldots + \lambda_{2+l}\mathbf{h}_{2+l} + \mathbf{x}^0, \\ \mathbf{x}^0 = \boldsymbol{\Delta}^{-1}\mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{b},\ \lambda_2 \in (0, \infty), \lambda_3 \in (0, \infty), \ldots, \lambda_{2+l} \in (0, \infty)\}. \end{aligned} \tag{25}$$

We observe the following properties of (25) as the key result of this paper:

- $\mathbf{\Delta}$ depends on λ_2 in (19). $\mathbf{h}_3 \ldots \mathbf{h}_{2+l}$ also depend on λ_2 in (24). Therefore, $\lambda_2 \ldots \lambda_{2+l}$ act as parameters of (25).
- With fixed λ_2 and varying $\lambda_3 \ldots \lambda_{2+l}$ (i.e., $\lambda_3 \in (0, \infty)$, …, $\lambda_{2+l} \in (0, \infty)$), (25) is a translated, open-ended pyramid. The pyramid is generated open-endedly by $\mathbf{h}_3 \ldots \mathbf{h}_{2+l}$ on the origin and then translated to $\mathbf{x}^0$. From now on, we mean all pyramids as translated and open-ended. $\mathbf{x}^0$ is the vertex of the pyramid and the minimum-variance portfolio of (15), or precisely, $\mathbf{x}^0$ is the optimal solution of the following weighted-sums model of quadratic objectives:

$$\begin{aligned} &\min \{\mathbf{x}^T \mathbf{\Sigma}_1 \mathbf{x} + \lambda_2 \mathbf{x}^T \mathbf{\Sigma}_2 \mathbf{x}\} \\ &\text{s.t. } \mathbf{A}^T \mathbf{x} = \mathbf{b} \end{aligned} \tag{26}$$

- Both (25) and $\mathbf{x}^0$ respectively extend their counterparts (8) and (7).
- By varying λ_2 (i.e., $\lambda_2 \in (0, \infty)$), we thus vary $\mathbf{\Delta}$ and $\mathbf{h}_3 \ldots \mathbf{h}_{2+l}$. With $\lambda_3 \in (0, \infty), \ldots, \lambda_{2+l} \in (0, \infty)$, (25) is then a sequence of pyramids.

We report the result in the following theorem.

Theorem 6 *The properly efficient set of (15) is (25). With fixed λ_2 and with $\lambda_3 \in (0, \infty), \ldots, \lambda_{2+l} \in (0, \infty)$, (25) is a pyramid. With varying λ_2 as $\lambda_2 \in (0, \infty)$ and with $\lambda_3 \in (0, \infty), \ldots, \lambda_{2+l} \in (0, \infty)$, (25) is a 1-dimensional sequence of pyramids.*

We present a partially hypothetical figure for two properly efficient sets and the corresponding sets in weighting vector space in Fig. 1. The term "hypothetical" is due to the fact that high-dimensional spaces which can not be directly depicted are involved in Fig. 1. We will present exact figures based on (true) historical data in Sect. 5.

In Panel A of Fig. 1, we depict weighting vector space $(1, \lambda_2, \lambda_3, \lambda_4)$ in the left. Three sets $\{\begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \in \mathbb{R}^4 \mid \lambda_3 \in (0, \infty), \lambda_4 \in (0, \infty)\}$ respectively with $\lambda_2 = 1$, 2, and 3 in the space are depicted as planes and marked by three different shades of gray. The three sets respectively map to three pyramids in the right of Panel A. The three pyramids are marked by the corresponding shades of gray and depicted as pyramids. We label the three sets and pyramids and mark the map by arrows. The curved path passing through the three pyramids' vertices demonstrates that there is a sequence of such pyramids.

In Panel B of Fig. 1, we depict weighting vector space $(1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ in the left. There are three sets $\{\begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \in \mathbb{R}^5 \mid \lambda_3 \in (0, \infty), \lambda_4 \in (0, \infty), \lambda_5 \in (0, \infty)\}$ respectively with $\lambda_2 = 1, 2, 3$ in a 5-dimensional weighting vector space. The sets are depicted as boxes and marked by three different shades of gray. Three correspondingly-mapped pyramids are in the right of Panel B.

With $\lambda_3 \in (0, \infty), \ldots, \lambda_{2+l} \in (0, \infty)$ in (25), the pyramids of (25) are not closed. Next, we discuss the dimensionality of such pyramids by computing the rank of matrix (24). An ideal situation is that the $n \times n$ matrix

$$\mathbf{\Delta}^{-1}(\mathbf{I}_n - \mathbf{A}(\mathbf{A}^T \mathbf{\Delta}^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{\Delta}^{-1}) \tag{27}$$

is invertible. Consequently, $rank(matrix(24)) = rank(\mathbf{C})$ and $\mathbf{h}_3 \ldots \mathbf{h}_{2+l}$ are linearly independent and the pyramids of (25) are l-dimensional because of Assumption 3. With $\mathbf{\Delta}$ as

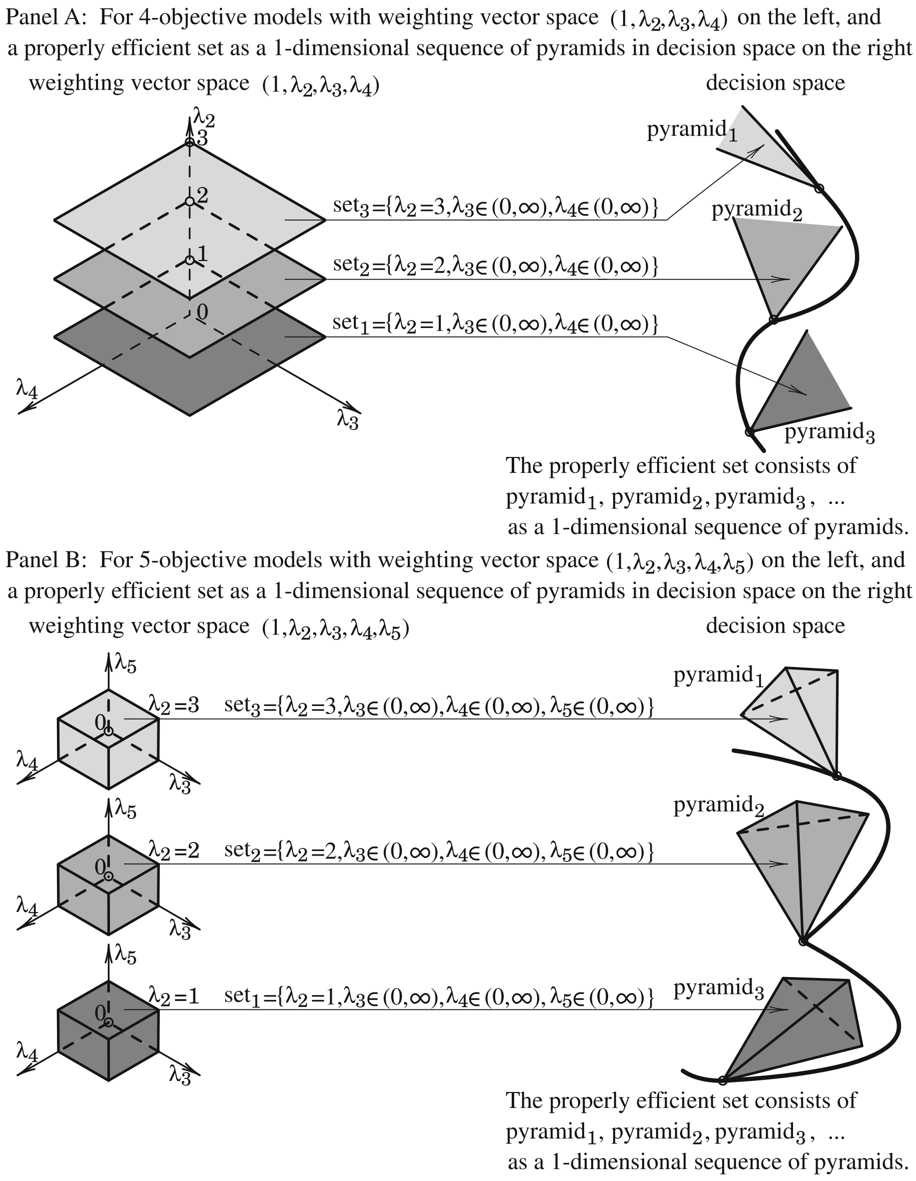

Fig. 1 Two properly efficient sets and the corresponding sets in weighting vector spaces

invertible, we premultiply (27) by $\mathbf{\Delta}$ and then postmultiply the product by $\mathbf{\Delta}$ and get

$$\mathbf{\Delta} - \mathbf{A}(\mathbf{A}^T\mathbf{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{A}^T. \tag{28}$$

Both matrix (28) and matrix (27) are possibly singular and thus negate the certain existence of the ideal situation by the following example: With $n = 4$, $k = 2$, and $m = 2$, $\mathbf{\Sigma}_1 = 0.5\mathbf{I}_4$ with

$\mathbf{I}_4$ as a 4×4 identity matrix, $\boldsymbol{\Sigma}_2 = \mathbf{I}_4$, $\lambda_2 = 0.5$, $\boldsymbol{\Delta} = \boldsymbol{\Sigma}_1 + \lambda_2 \boldsymbol{\Sigma}_2 = \mathbf{I}_4$, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, matrix (28)=matrix (27) $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ as singular. Therefore, the dimensionality of the pyramids of (25) can depend on problems. Next, we substitute (25) into the objectives of (15) to get the properly nondominated surface of (15) as

$$\left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_{2+l} \end{bmatrix} \in \mathbb{R}^{2+l} \mid z_1 = (\lambda_3 \mathbf{h}_3 + \ldots + \lambda_{2+l} \mathbf{h}_{2+l} + \mathbf{x}^0)^T \boldsymbol{\Sigma}_1 (\lambda_3 \mathbf{h}_3 + \ldots + \lambda_{2+l} \mathbf{h}_{2+l} + \mathbf{x}^0), \right.$$
$$z_2 = (\lambda_3 \mathbf{h}_3 + \ldots + \lambda_{2+l} \mathbf{h}_{2+l} + \mathbf{x}^0)^T \boldsymbol{\Sigma}_2 (\lambda_3 \mathbf{h}_3 + \ldots + \lambda_{2+l} \mathbf{h}_{2+l} + \mathbf{x}^0),$$
$$\left. \begin{bmatrix} z_3 \\ \vdots \\ z_{2+l} \end{bmatrix} = \mathbf{C}^T (\lambda_3 \mathbf{h}_3 + \ldots + \lambda_{2+l} \mathbf{h}_{2+l} + \mathbf{x}^0),\ \lambda_2 \in (0, \infty), \ldots, \lambda_{2+l} \in (0, \infty) \right\}. \tag{29}$$

Next, we compute the weakly efficient set of (15) by studying (16) with $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix} \gneq \mathbf{0}^{2+l}$ (where $\mathbf{0}^{2+l}$ is a $(2+l) \times 1$ vector of zeros, $\mathbf{0}^l$ is an $l \times 1$ vector of zeros, and $\mathbf{0}^2$ is a 2×1 vector of zeros). The set $\{ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix} \gneq \mathbf{0}^{2+l} \}$ can be exhaustively and mutually exclusively divided into the following cases:

$$\left\{ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix} \gneq \mathbf{0}^{2+l} \right\} = \left\{ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix} > \mathbf{0}^{2+l} \right\} \cup \left\{ \begin{bmatrix} \lambda_1 > 0 \\ \lambda_2 = 0 \\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} \lambda_1 = 0 \\ \lambda_2 > 0 \\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l \end{bmatrix} \right\} \cup$$
$$\left\{ \begin{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \gneq \mathbf{0}^2 \\ \boldsymbol{\lambda}_3 = \mathbf{0}^l \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{0}^2 \\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l \end{bmatrix} \right\}. \tag{30}$$

Then by Theorem 5, we demonstrate that the weakly efficient set of (15) can be computed by the union of all optimal solutions of (16) with all the cases in the following theorem:

Theorem 7 *The weakly efficient set of (15) equals the union of all optimal solutions of (16) with the following cases:*

(a) For the case $\left\{ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \boldsymbol{\lambda}_3 \end{bmatrix} > \mathbf{0}^{2+l} \right\}$ *for (16), the optimal solutions are specified in (25) and Theorem 6.*

(b) For the case $\left\{ \begin{bmatrix} \lambda_1 > 0 \\ \lambda_2 = 0 \\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l \end{bmatrix} \right\}$ *for (16),* $\boldsymbol{\Delta} = \boldsymbol{\Sigma}_1 + 0\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_1$ *and does not depend on* λ_2 *in (19). The optimal solutions*

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \lambda_3 \mathbf{h}_3 + \ldots + \lambda_{2+l} \mathbf{h}_{2+l} + \mathbf{x}^1, \lambda_3 \in [0, \infty), \ldots, \lambda_{2+l} \in [0, \infty)\}.$$

are a closed pyramid.

(c) For the case $\left\{\begin{bmatrix}\lambda_1 = 0\\ \lambda_2 > 0\\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l\end{bmatrix}\right\}$ *for (16), the analyses from (17) to (28) do not hold because* λ_1*=0. However, we can exchange the order of* $z_1 = \mathbf{x}^T\boldsymbol{\Sigma}_1\mathbf{x}$ *and* $z_2 = \mathbf{x}^T\boldsymbol{\Sigma}_2\mathbf{x}$ *in (15). The analysis for (b) holds, and the optimal solutions are a closed pyramid.*

(d) For the case $\left\{\begin{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \gneq \mathbf{0}^2\\ \boldsymbol{\lambda}_3 = \mathbf{0}^l\end{bmatrix}\right\}$ *for (16), the pyramids of (25) degenerate into points. The optimal solutions are a sequence of points.*

(e) For the case $\left\{\begin{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \mathbf{0}^2\\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l\end{bmatrix}\right\}$ *for (16), the optimal solutions are an empty set (i.e., (16) has no optimal solution). We prove the result in the lemma below.*

Lemma 3 *With* $\left\{\begin{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \mathbf{0}^2\\ \boldsymbol{\lambda}_3 \gneq \mathbf{0}^l\end{bmatrix}\right\}$, *(16) becomes as follows:*

$$\begin{aligned}&max\,\{\boldsymbol{\lambda}_3^T\mathbf{C}^T\mathbf{x}\}\\ &s.t.\ \mathbf{A}^T\mathbf{x} = \mathbf{b}\end{aligned} \tag{31}$$

(31) does not have an upper bound (i.e., (31) has no optimal solution).

Proof By Assumption 3, we can partition $\mathbf{A}$ into $\mathbf{A} = \left[\frac{\mathbf{B}}{\mathbf{N}}\right]$ where $\mathbf{B}$ is $m \times m$ and invertible, and $\mathbf{N}$ is $(n-m)\times m$.[6] Similarly, we respectively partition $\mathbf{C}$ and $\mathbf{x}$ into $\mathbf{C} = \left[\frac{\mathbf{C}_B}{\mathbf{C}_N}\right]$ and $\mathbf{x} = \left[\frac{\mathbf{x}_B}{\mathbf{x}_N}\right]$ where $\mathbf{C}_B$ is $m \times l$, $\mathbf{C}_N$ is $(n-m)\times l$, $\mathbf{x}_B = \begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}$, and $\mathbf{x}_N = \begin{bmatrix}x_{m+1}\\ \vdots\\ x_n\end{bmatrix}$.

Then, $\mathbf{A}^T\mathbf{x} = \mathbf{b}$ is rewritten as $\left[\mathbf{B}^T \middle| \mathbf{N}^T\right]\left[\frac{\mathbf{x}_B}{\mathbf{x}_N}\right] = \mathbf{b}$ and $\mathbf{B}^T\mathbf{x}_B + \mathbf{N}^T\mathbf{x}_N = \mathbf{b}$. We premultiply $\mathbf{B}^T\mathbf{x}_B + \mathbf{N}^T\mathbf{x}_N = \mathbf{b}$ by $(\mathbf{B}^T)^{-1}$, rearrange the product, and get $\mathbf{x}_B = (\mathbf{B}^T)^{-1}\mathbf{b} - (\mathbf{B}^T)^{-1}\mathbf{N}^T\mathbf{x}_N$. Then, we substitute $\mathbf{x}_B$ into $\mathbf{C}^T\mathbf{x} = \left[\mathbf{C}_B^T \middle| \mathbf{C}_N^T\right]\left[\frac{\mathbf{x}_B}{\mathbf{x}_N}\right]$ and get

$$\mathbf{C}^T\mathbf{x} = \mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{b} + (\mathbf{C}_N^T - \mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T)\mathbf{x}_N. \tag{32}$$

With $\mathbf{A} = \left[\frac{\mathbf{B}}{\mathbf{N}}\right]$ and $\mathbf{C} = \left[\frac{\mathbf{C}_B}{\mathbf{C}_N}\right]$, $\left[\mathbf{A}\middle|\mathbf{C}\right]^T = \left[\begin{array}{c|c}\mathbf{B}^T & \mathbf{N}^T\\ \hline \mathbf{C}_B^T & \mathbf{C}_N^T\end{array}\right]$. Then, we compute $\left[\begin{array}{c|c}\mathbf{I}_B^{m\times m} & \mathbf{0}^{m\times l}\\ \hline -\mathbf{C}_B^{T\,l\times m} & \mathbf{I}_N^{l\times l}\end{array}\right]\left[\begin{array}{c|c}(\mathbf{B}^T)^{-1\,m\times m} & \mathbf{0}^{m\times l}\\ \hline \mathbf{0}^{l\times m} & \mathbf{I}_N^{l\times l}\end{array}\right]\left[\begin{array}{c|c}\mathbf{B}^{T\,m\times m} & \mathbf{N}^{T\,m\times(n-m)}\\ \hline \mathbf{C}_B^{T\,l\times m} & \mathbf{C}_N^{T\,l\times(n-m)}\end{array}\right]$ where $\mathbf{I}_B$ is an identity matrix, $\mathbf{0}$ is a matrix of zeros, $\mathbf{I}_N$ is an identity matrix, and all the matrices' dimensions are

[6] By Assumption 3, there exists an $m \times m$ and invertible submatrix of $\mathbf{A}$, and we can take the first m rows of $\mathbf{A}$ as the submatrix. Otherwise, we can exchange the rows of $\mathbf{A}$ to make the first m rows of $\mathbf{A}$ as the submatrix and exchange the corresponding parts of $\mathbf{x}$, $\mathbf{C}$, etc.

written as superscripts.

$$\left[\begin{array}{c|c}\mathbf{I}_B & \mathbf{0}\\ \hline -\mathbf{C}_B^T & \mathbf{I}_N\end{array}\right]\left[\begin{array}{c|c}(\mathbf{B}^T)^{-1} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{I}_N\end{array}\right]\left[\begin{array}{c|c}\mathbf{B}^T & \mathbf{N}^T\\ \hline \mathbf{C}_B^T & \mathbf{C}_N^T\end{array}\right]=\left[\begin{array}{c|c}\mathbf{I}_B & \mathbf{0}\\ \hline -\mathbf{C}_B^T & \mathbf{I}_N\end{array}\right]\left[\begin{array}{c|c}\mathbf{I}_B & (\mathbf{B}^T)^{-1}\mathbf{N}^T\\ \hline \mathbf{C}_B^T & \mathbf{C}_N^T\end{array}\right]$$
$$\left[\begin{array}{c|c}\mathbf{I}_B & \mathbf{0}\\ \hline -\mathbf{C}_B^T & \mathbf{I}_N\end{array}\right]\left[\begin{array}{c|c}(\mathbf{B}^T)^{-1} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{I}_N\end{array}\right]\left[\begin{array}{c|c}\mathbf{B}^T & \mathbf{N}^T\\ \hline \mathbf{C}_B^T & \mathbf{C}_N^T\end{array}\right]=\left[\begin{array}{c|c}\mathbf{I}_B & (\mathbf{B}^T)^{-1}\mathbf{N}^T\\ \hline \mathbf{0} & \mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T\end{array}\right] \tag{33}$$

Because $\mathbf{I}_B$, $\mathbf{I}_N$, and $(\mathbf{B}^T)^{-1}$ are all invertible, both $\left[\begin{array}{c|c}\mathbf{I}_B & \mathbf{0}\\ \hline -\mathbf{C}_B^T & \mathbf{I}_N\end{array}\right]$ and $\left[\begin{array}{c|c}(\mathbf{B}^T)^{-1} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{I}_N\end{array}\right]$ are invertible by Lax (2007). Therefore, $rank(\left[\begin{array}{c|c}\mathbf{B}^T & \mathbf{N}^T\\ \hline \mathbf{C}_B^T & \mathbf{C}_N^T\end{array}\right])=rank(\left[\begin{array}{c|c}\mathbf{I}_B & (\mathbf{B}^T)^{-1}\mathbf{N}^T\\ \hline \mathbf{0} & \mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T\end{array}\right])$ by (33). By Assumption 3 and $\left[\mathbf{A}|\mathbf{C}\right]^T=\left[\begin{array}{c|c}\mathbf{B}^T & \mathbf{N}^T\\ \hline \mathbf{C}_B^T & \mathbf{C}_N^T\end{array}\right]$, $rank(\left[\begin{array}{c|c}\mathbf{I}_B & (\mathbf{B}^T)^{-1}\mathbf{N}^T\\ \hline \mathbf{0} & \mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T\end{array}\right])=m+l$. Then with $\mathbf{I}_B$ as invertible, $rank(\mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T)=l$ (i.e., $(\mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T)^{l\times(n-m)}$ has full row rank) by Lax (2007). The rows of $\mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T$ are linearly independent.

For $\boldsymbol{\lambda}_3 \gneq \mathbf{0}^l$, by (32), we compute the following:

$$\boldsymbol{\lambda}_3^T\mathbf{C}^T\mathbf{x}=\boldsymbol{\lambda}_3^T\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{b}+\boldsymbol{\lambda}_3^T(\mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T)\mathbf{x}_N \tag{34}$$

We obtain $\boldsymbol{\lambda}_3^T(\mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T)\neq\mathbf{0}^{n-m}$ (where $\mathbf{0}^{n-m}$ is a $1\times(n-m)$ vector of zeros) because of the linear independence. Suppose that the first element of $\boldsymbol{\lambda}_3^T(\mathbf{C}_N^T-\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{N}^T)\neq\mathbf{0}^{n-m}$ is $g>0$. We take $\bar{\mathbf{x}}_N=\begin{bmatrix}t\\0\\\vdots\\0\end{bmatrix}$ with $t\in\mathbb{R}$ and $\bar{\mathbf{x}}_B=(\mathbf{B}^T)^{-1}\mathbf{b}-(\mathbf{B}^T)^{-1}\mathbf{N}^T\bar{\mathbf{x}}_N$. By (34), we obtain $\boldsymbol{\lambda}_3^T\mathbf{C}^T\bar{\mathbf{x}}=\boldsymbol{\lambda}_3^T\mathbf{C}_B^T(\mathbf{B}^T)^{-1}\mathbf{b}+gt\to\infty$ as $t\to\infty$. Therefore, (31) does not have an upper bound. □

Then, the efficient set of (15) is between the properly efficient set specified by Theorem 6 and the weakly efficient set specified by Theorem 7. We have also tried to analyze (15) by an e-constraint method. Unfortunately, a system of nonlinear equations is encountered in the analysis.

5 An illustration

In this section for three stocks, we illustrate the properly efficient set and properly nondominated set of the following model:

$$\begin{aligned}\min\{z_1&=\mathbf{x}^T\boldsymbol{\Sigma}_1\mathbf{x}\}\\ \min\{z_2&=\mathbf{x}^T\boldsymbol{\Sigma}_2\mathbf{x}\}\\ \max\{z_3&=\mathbf{c}_1\mathbf{x}\}\\ \text{s.t. }\mathbf{1}^T\mathbf{x}&=1\end{aligned} \tag{35}$$

We purposefully choose three stocks in order to directly depict the properly efficient set in 3-dimensional decision space. We also purposefully set three objectives in order to directly depict the properly nondominated set in 3-dimensional criterion space. Moreover, $\mathbf{A}=\mathbf{1}$,

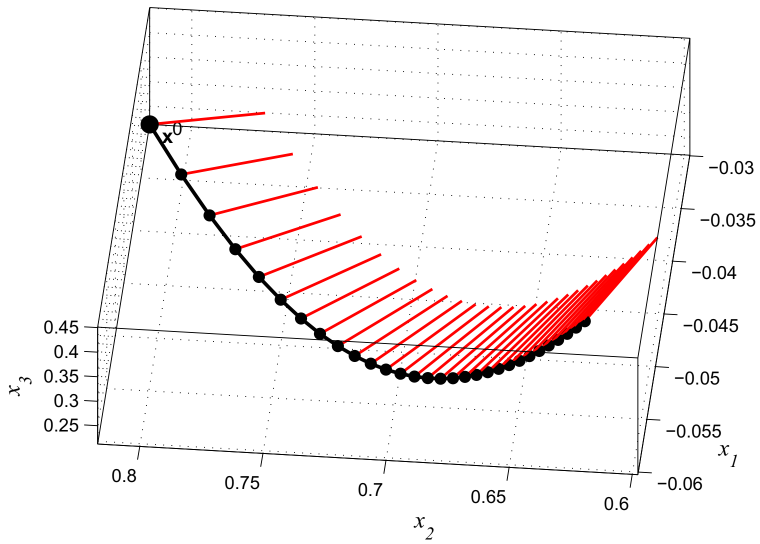

Fig. 2 The properly efficient set of (35) in decision space

so S is a 2-dimensional affine subspace and we can see the properly efficient set's structure. Otherwise, if $\mathbf{A}$ is 3 by 2, S can be 1-dimensional. The properly efficient set is compressed into 1-dimension and is not clearly visible. For (35), z_1 and z_3 respectively measure the variance of portfolio return and expected portfolio return, and z_2 measures the variance of portfolio liquidity.

We choose American Express Co. (AXP) and Coca Cola Co. (KO) and Disney Walt Co. (DIS), sample the three stocks' monthly returns from January 2008 to December 2012, and respectively take the sample covariance matrix and sample mean as $\boldsymbol{\Sigma}_1$ and $\mathbf{c}_1$. [7] We also sample the stocks' monthly bid-asked spread ratio ($\frac{\text{bid-asked spread}}{\text{price}}$) to measure liquidity in the same time period and take the sample covariance matrix and sample mean as $\boldsymbol{\Sigma}_2$ as follows:

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 0.0219 & 0.0019 & 0.0072 \\ 0.0019 & 0.0026 & 0.0017 \\ 0.0072 & 0.0017 & 0.0059 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 7.6779E-07 & 3.7496E-07 & -4.8947E-08 \\ 3.7496E-07 & 4.1621E-07 & 3.1361E-08 \\ -4.8947E-08 & 3.1361E-08 & 3.0532E-07 \end{bmatrix}$$

$$\mathbf{c}_1 = \begin{bmatrix} 0.0123 & 0.0066 & 0.0114 \end{bmatrix}$$

We delineate the properly efficient set of (35) in Fig. 2. The cone vertices are marked by black dots. The cone generators are depicted as red (printed as gray) lines. By connecting the dots with a curve, we see the properly efficient set as a sequence of cones. We mark the minimum-variance portfolio by a bigger black dot and label the portfolio as $\mathbf{x}^0$.

We delineate the properly nondominated set of (35) in Fig. 3 and mark the criterion vector of $\mathbf{x}^0$ as $\mathbf{z}^0$. The criterion vectors of each cone are a curve in criterion space. We can view

[7] Data source: Wharton Research Data Services (WRDS), ⟨https://wrds-web.wharton.upenn.edu/wrds/⟩, July 2, 2013.

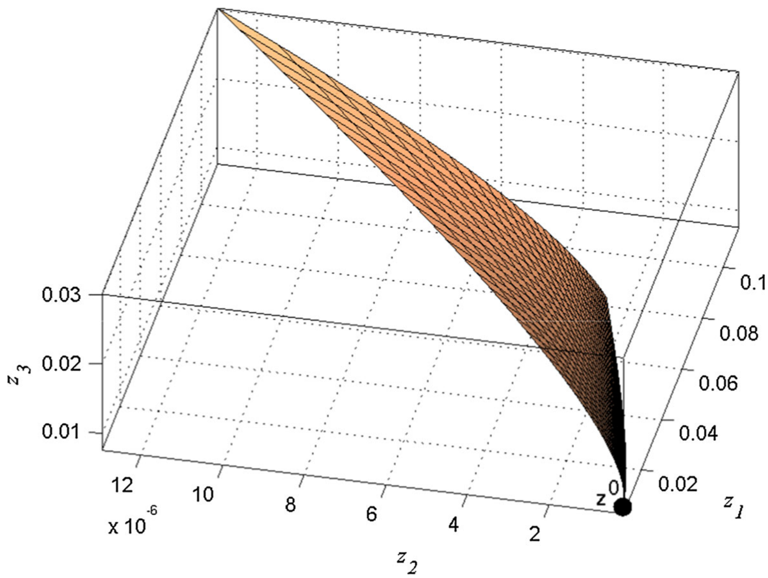

Fig. 3 The properly nondominated set of (35) in criterion space

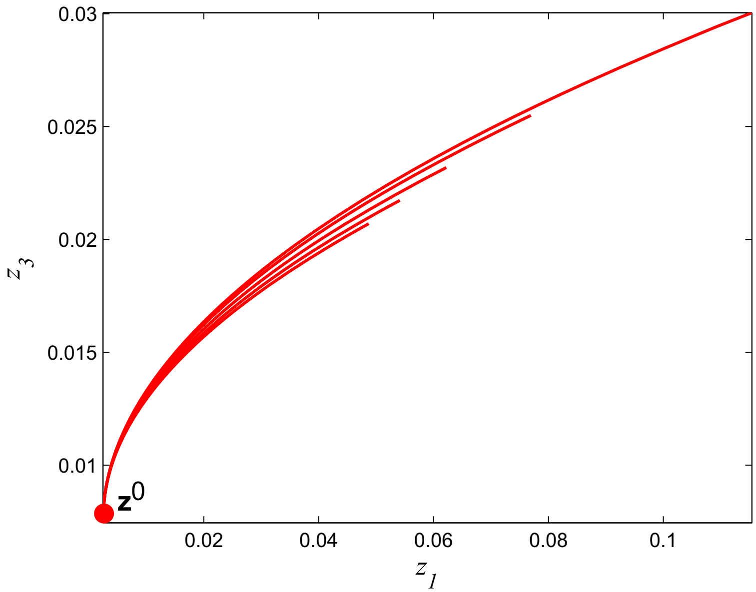

Fig. 4 The projection of the properly nondominated set of (35) onto *(variance of portfolio return, expected portfolio return)* space

the properly nondominated set as a sequence of such curves. We delineate the projection of the properly nondominated set as a sequence of such curves in Fig. 4.

6 Extending to general *k*-quadratic objective models

We extend (15) into a model with k-quadratic objectives and also extend the theorems in Sects. 3 and 4 without proofs because the proofs are of direct extension of their counterparts in Sects. 3 and 4. We introduce r_2 in (10) and $\mathbf{\Sigma}_2$ and $\boldsymbol{\mu}_2$ in (11). We study as follows:

$$\begin{aligned}
&\min\{z_1 = var(r_1) = \mathbf{x}^T\mathbf{\Sigma}_1\mathbf{x}\}\\
&\qquad\vdots\\
&\min\{z_k = var(r_k) = \mathbf{x}^T\mathbf{\Sigma}_k\mathbf{x}\}\\
&\max\{z_{k+1} = E(r_1) = \boldsymbol{\mu}_1{}^T\mathbf{x}\}\\
&\qquad\vdots\\
&\max\{z_{2k} = E(r_k) = \boldsymbol{\mu}_k{}^T\mathbf{x}\}\\
&\quad\text{s.t. } \mathbf{x} \in S
\end{aligned} \tag{36}$$

By the research of Caballero et al. (2001), (36) is a variance-expectation operationalization of the following stochastic multiple-objective programming model:

$$\begin{aligned}
&\max\{r_1\}\\
&\quad\vdots\\
&\max\{r_k\}\\
&\quad\text{s.t. } \mathbf{x} \in S.
\end{aligned}$$

Suppose that the utility function of r_1 is $u_1(r_1) = r_1 - \frac{1}{2}q_1r_1^2$ with $q_1 > 0$, the utility function of r_k is $u_k(r_k) = r_k - \frac{1}{2}q_kr_k^2$ with $q_k > 0$, and the extended utility function of $(r_1, \ldots, r_k)$ is

$$u(r_1, \ldots, r_k) = u_1(r_1) + \ldots + u_k(r_k) = r_1 - \frac{1}{2}q_1r_1^2 + \ldots + r_k - \frac{1}{2}q_kr_k^2.$$

Portfolio selection can be formulated by the extended utility function as

$$\begin{aligned}
&\max\{E(u(r_1, \ldots, r_k))\}\\
&\qquad\text{s.t. } \mathbf{x} \in S.
\end{aligned} \tag{37}$$

Then, we justify (36) by (37) in the following theorem:

Theorem 8 *For $u(r_1, \ldots, r_k)$, if $r_1 \le \frac{1}{q_1} \ldots r_k \le \frac{1}{q_k}$, the optimal solution of (37) is an efficient solution of (36).*

Then, we further extend (36) into the following model:

$$\begin{aligned}
&\min\{z_1 = \mathbf{x}^T\mathbf{\Sigma}_1\mathbf{x}\}\\
&\qquad\vdots\\
&\min\{z_k = \mathbf{x}^T\mathbf{\Sigma}_k\mathbf{x}\}\\
&\max\{z_{k+1} = \mathbf{c}_1{}^T\mathbf{x}\}\\
&\qquad\vdots\\
&\max\{z_{k+l} = \mathbf{c}_l{}^T\mathbf{x}\}
\end{aligned}$$

$$\text{s.t. } \mathbf{A}^T\mathbf{x} = \mathbf{b} \tag{38}$$

We apply a weighted-sums method to (38) and get

$$\begin{aligned} \min \{\lambda_1 \mathbf{x}^T \boldsymbol{\Sigma}_1 \mathbf{x} + \cdots + \lambda_k \mathbf{x}^T \boldsymbol{\Sigma}_k \mathbf{x} - \boldsymbol{\lambda}_3^T \mathbf{C}^T \mathbf{x}\} \\ \text{s.t. } \mathbf{A}^T\mathbf{x} = \mathbf{b} \end{aligned} \tag{39}$$

where $\boldsymbol{\lambda}_3 = \begin{bmatrix} \lambda_{k+1} \\ \vdots \\ \lambda_{k+l} \end{bmatrix}$, and $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ \boldsymbol{\lambda}_3 \end{bmatrix}$ is a weighting vector for (38). We denote the properly efficient set of (38) as $PE(38)$. Then, we prove that $PE(38)$ can be obtained by all the optimal solutions of (39) with all $\boldsymbol{\lambda} > \mathbf{0}$ (where $\mathbf{0}$ is a $(k+l) \times 1$ vector of zeros) in the following theorem:

Theorem 9 *$PE(38)$={all the optimal solutions of (39) with all $\boldsymbol{\lambda} > \mathbf{0}$}.*

Basically following the analyses from (16) to (25), $PE(38)$ is

$$\begin{aligned} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \lambda_{k+1}\mathbf{h}_3 + \ldots + \lambda_{k+l}\mathbf{h}_{2+l} + \mathbf{x}^0, \\ \mathbf{x}^0 = \boldsymbol{\Delta}^{-1}\mathbf{A}(\mathbf{A}^T\boldsymbol{\Delta}^{-1}\mathbf{A})^{-1}\mathbf{b},\ \lambda_2 \in (0,\infty), \lambda_{k+1} \in (0,\infty), \ldots, \lambda_{k+l} \in (0,\infty)\} \end{aligned} \tag{40}$$

where $\boldsymbol{\Delta} = \boldsymbol{\Sigma}_1 + \lambda_2\boldsymbol{\Sigma}_2 + \ldots + \lambda_k\boldsymbol{\Sigma}_k$. $\mathbf{x}^0$ is the vertex of the pyramid and the minimum-variance portfolio of (38), or more precisely, $\mathbf{x}^0$ is the optimal solution of the following weighted-sums model of quadratic objectives:

$$\begin{aligned} \min \{\mathbf{x}^T \boldsymbol{\Sigma}_1 \mathbf{x} + \lambda_2 \mathbf{x}^T \boldsymbol{\Sigma}_2 \mathbf{x} + \ldots + \lambda_k \mathbf{x}^T \boldsymbol{\Sigma}_k \mathbf{x}\} \\ \text{s.t. } \mathbf{A}^T\mathbf{x} = \mathbf{b}. \end{aligned} \tag{41}$$

Theorem 10 *The properly efficient set of (38) is (40). With fixed $\lambda_2 \ldots \lambda_k$ and with $\lambda_{k+1} \in (0,\infty) \ldots \lambda_{k+l} \in (0,\infty)$, (40) is a pyramid. With varying $\lambda_2 \ldots \lambda_k$ as $\lambda_2 \in (0,\infty) \ldots \lambda_k \in (0,\infty)$ and with $\lambda_{k+1} \in (0,\infty) \ldots \lambda_{k+l} \in (0,\infty)$, (40) is $(k-1)$-dimensional sequences of pyramids.*

Then, we substitute (40) into (38) to get the properly nondominated set.

In the left of Fig. 5, we depict six sets $\left\{ \begin{bmatrix} 1 \\ \lambda_2 \\ \vdots \\ \lambda_6 \end{bmatrix} \in \mathbb{R}^6 \mid \lambda_4 \in (0,\infty) \ldots \lambda_6 \in (0,\infty) \right\}$ respectively with $\lambda_2 = 1, 2$, and 3 and $\lambda_3 = 1$ and 2 in a 6-dimensional weighting space. The six sets are illustrated as boxes and marked by three different shades of gray according to λ_2 =1, 2, and 3. The six sets respectively map to six pyramids in the right. The six pyramids are marked by the corresponding shades of gray. According to $\lambda_3 = 1$ and 2, two curved paths pass through the six pyramids' vertices and demonstrate that there are 2-dimensional (according to λ_2 and λ_3) sequences of such pyramids.

$\left\{ \begin{bmatrix} 1 \\ \lambda_2 \\ \vdots \\ \lambda_6 \end{bmatrix} \in \mathbb{R}^6 \mid \lambda_4 \in (0,\infty) \ldots \lambda_6 \in (0,\infty) \right\}$ respectively with $\lambda_2 = 1, 2$, and 3 and $\lambda_3 = 1$ and 2 in a 6-dimensional weighting vector space in the left of Fig. 5. The six sets are

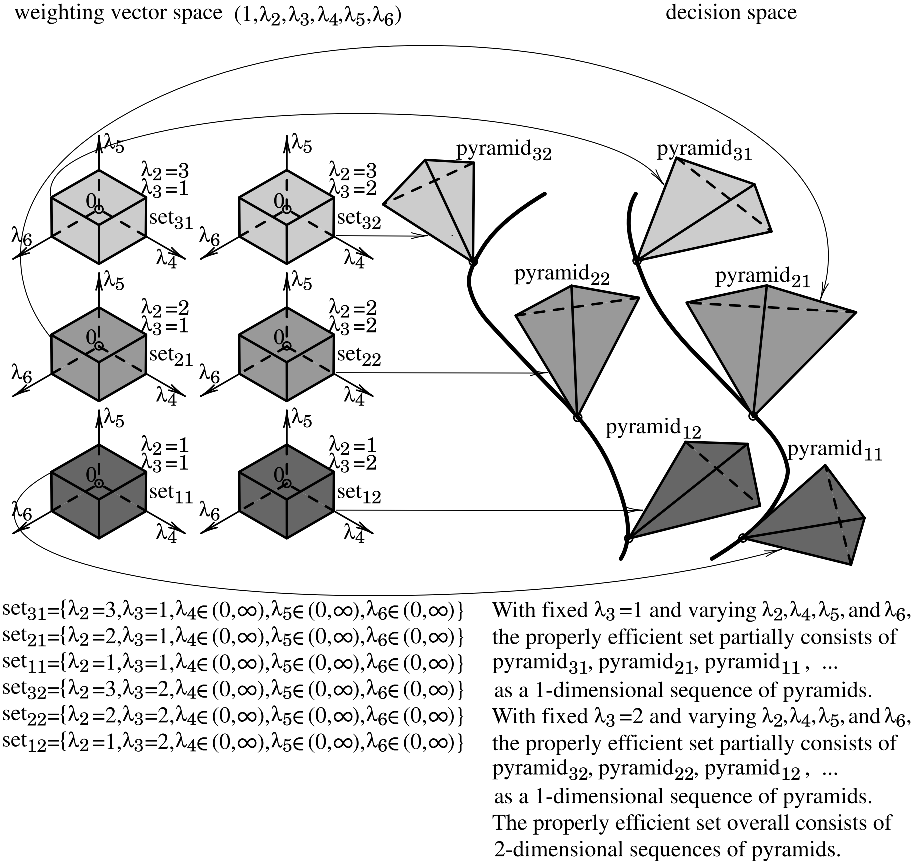

Fig. 5 A properly efficient set and the corresponding sets in weighting vector space

illustrated as boxes and marked by three different shades of gray according to λ_2 =1, 2, and 3. The six sets respectively map to six pyramids in the right. The six pyramids are marked by the corresponding shades of gray.

7 Conclusion

The purpose of this paper is to make analytical derivation progress on multiple criteria (meaning three or more objectives) portfolio selection problems in which two of the objectives are quadratic. While analytical derivation achievements have been carried out in complete form by Merton (1972) and Roll (1977) for the bi-criterion case in which there is only one quadratic objective, this paper is the first analytical derivation paper of which we are aware to address portfolio problems in which two of the objectives are quadratic, as for instance, with short-term variance and long-term variance in Garcia-Bernabeu et al. (2019). But because of the more difficult mathematics involved, it is anticipated that it will take much more than five years and two papers as it did with Merton and Roll to complete the multiple objective case with two quadratic objectives. Thus, it is along these lines that we see many opportunities for research ahead of us to complete the task in this area.

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Availability of data and materials We have already deposited the data and materials in Mendeley Data https://data.mendeley.com/datasets/df59c5vc36/1

Declarations

Conflict of interest The author declare that they have no Conflict of interest.

Code availability. We have already deposited the code in Mendeley Data https://data.mendeley.com/datasets/df59c5vc36/1

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