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Multiple Objectives in Portfolio Selection

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Abstract

We begin this paper by first comparing the theory of present-day portfolio selection, which is a theory for *standard* investors (whose utility functions take on only the single argument of portfolio return), with a developing theory for *non-standard* investors (whose utility functions are allowed to take on additional arguments). Examples of additional arguments are dividends, liquidity, social responsibility, amount of short selling, and so forth. Then, with portfolio selection for non-standard investors taking on the form of a multi-objective stochastic programming problem, equivalent deterministic formulations involving more than mean and variance are explored. With the nondominated sets of non-standard investors no longer frontiers, but now surfaces, the tools and techniques that must be imported from multiple criteria optimization to compute and analyze them are next discussed. The paper concludes with a list of research topics that are candidates for extending the multiple criteria portfolio selection material of this paper.

Keywords: Multiple criteria portfolio selection, multi-objective stochastic programming, equivalent deterministic problems, parametric quadratic programming, nondominated surfaces, hyperboloidic platelets.

1. Introduction

In finance there has always been the problem of how to combine investments to form a portfolio. Progress on this problem, called *portfolio selection*, did not reach a historical juncture until the 1950s. During that decade, Markowitz (1952) developed his mean-variance formulation, introduced an algorithm (Markowitz, 1956) for solving for the "efficient frontier", and advised on the selection of one's optimal portfolio from the efficient frontier. Also, there was Roy's method (Roy, 1952) for the direct computation of one's "safety first" portfolio. At the intersection of finance and operations research, their work not only provided formal methods for solving portfolio problems, but the influence of their work gave impetus to the topic of quadratic programming in operations research and to the branch of finance known today as "modern portfolio analysis". With regard to books, one could mention that Schrage (2003) provides excellent applications of quadratic programming to problems in portfolio selection and that Elton et al. (2002) is often considered the canonical text on modern portfolio analysis.

Since the mean-variance approaches of Markowitz and Roy, hundreds if not thousands of papers have been published on portfolio selection. Despite scrutiny of what might appear to be every imaginable aspect of portfolio selection, two assumptions have endured essentially unchanged. One is that the purpose of portfolio selection is wealth maximization, i.e., to only make money (the more the better). The other is that in

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pursuing this purpose, it is only necessary to monitor at the utility function level one's portfolio with one random variable, that being the random variable of portfolio return. Because these assumptions have been the "standard" for over 50 years, and because virtually all of what is known today about portfolio selection is based upon them, we will call this body of knowledge *standard portfolio* theory, and all investors who feel that the assumptions apply to them *standard* investors.

If you are a standard investor, you have been well served by enormous amounts of research. But if you are a *non-standard* investor, you do not have much of a road map. In response, the purpose of this paper is to fill in some of the gaps for this largely overlooked, but potentially very important and growing, class of investors. How might it be that the two assumptions might not reflect you? There are at least two ways. One way is that you are in portfolio selection for more than the money. For example, you also want your portfolio to help the environment, encourage good corporate governance, and contain the smallest number of securities whereby all of your intentions can be satisfied to the greatest extent possible. In this case, you would want to be monitoring your portfolio at the utility function level with regard to four arguments. Supporters of standard portfolio theory could probably be counted on to claim that all arguments beyond portfolio return are unnecessary. They might well say that, because of efficient markets, all appropriate concerns about the environment and corporate governance are already in the prices of securities, so no extra treatment is necessary on their account. But what if you do not believe that markets are efficient all the time? As for the smallest number of securities, you might well be told that this concern should be modeled as a requirement (i.e., constraint) with an appropriate right-hand side. However, knowing how to set the right-hand side values of such constraints is not easy without being in a position to have knowledge of the trade-offs (which this approach does not facilitate), so this technique is less than ideal.

A second way in which one might differ from the standard assumptions is as follows. Let it be true that you are only in portfolio selection for the money. But let it also be true that you do not fully trust the data used to characterize the portfolio return random variable. To be on the safe side, it would not be unreasonable for you to be interested in the monitoring of your portfolio with a utility function that accepts other corroborating arguments such as dividends, amount invested in R&D, growth in sales, and so forth.

Perhaps one of the reasons for the endurance of the two assumptions is that for a long time portfolio selection had no recourse. There were no techniques for analyzing portfolio problems with multiple objectives, i.e., with more than one utility function argument associated with them. But with the area of multiple criteria optimization now well developed, it is now possible to begin integrating applicable methods and procedures from multiple criteria optimization into portfolio selection to better serve the needs of non-standard investors. In this way, maybe the research support of non-standard investors can someday catch up with that of standard investors.

Despite the volume of research supporting standard portfolio selection, there has always been a slight undercurrent of multiple objectives in portfolio selection, but this undercurrent has become more pronounced in recent years. To gain a perspective on the literature of multiple objectives in portfolio selection and how it relates to this paper, we utilize seven categories. The first three categories are overview pieces, algorithmic articles for characterizing the nondominated set, and skewness papers. The next four result from classifying the remaining papers as to whether they are methodological or application, and as to whether they provide for a direct computation of an optimal portfolio or involve some sort of interactive search of the nondominated set before terminating at a final solution.

In the overview category is Hallerbach and Spronk (2002) in which the benefits of incorporating the techniques of multiple criteria analysis into financial decision making in general are highlighted. Other relevant overview contributions are by Pardalos et al. (1994), Spronk and Hallerbach (1997), Bana e Costa and Soares (2004), and Steuer et al. (2005). In the second category we mention two papers. For portfolio problems in which there are additional linear objectives, Fliege (2004) develops an ingenious procedure for approximating the nondominated set to any degree of resolution with evenly dispersed points. Also, there is

Streichert et al. (2003) in which they use evolutionary algorithms to compute families of discretized efficient frontiers as a function of the number of securities in a portfolio. In the third category we have skewness papers as exemplified by Stone (1973), Konno et al. (1993), Konno and Suzuki (1995), Chunhachinda et al. (1997), and Prakash et al. (2003). Skewness is given its own category as there can be debate about its criterion status. We follow up on this in Section 2.

In the methodological/direct computation category, where the attempt is to develop theoretical procedures for the incorporation of decision-maker preferences into a model to enable the direct computation of an "optimal" portfolio, we have Ballestero (1998) and Ogryczak (2000). In the application/direct computation category, where the same thing is done but in more of the context of an application, we have Aouni et al. (2004), L'Hoir and Teghem (1995), and Ehrgott et al. (2004).

In the methodological/interactive search category, where the attempt is to develop theoretical procedures for iteratively searching the nondominated set, we have Korhonen and Yu (1997, 1998), Xu and Li (2002), and Kliber (2005). In the application/interactive search category we have for example Chow (1995), Lo et al. (2003), and Tamiz et al. (1996).

The rest of the paper is organized as follows. In Section 2 we show portfolio selection in the light of a single-criterion stochastic programming problem and discuss the idea of obtaining equivalent deterministic problem formulations for solution. In Section 3 we augment the single-criterion stochastic programming program of the previous section with additional stochastic and deterministic objectives and further discuss equivalent deterministic problems in this more complex situation. In Section 4 we discuss the different ways people from traditionally-trained finance and multiple criteria optimization might view portfolio selection in a mathematical programming context. In Section 5 we discuss the nature of the efficient and nondominated sets when additional linear criteria are present in an equivalent deterministic problem. In Section 6 we discuss how it is no longer necessary to artificially diagonalize the covariance matrix structure in order to make large-scale portfolio optimization problems amenable to computation. In Section 7 we enumerate research topics in need of investigation to further support developments in multiple criteria portfolio selection.

2. Stochastic versus Deterministic Sides

We begin this section by stating that it is a rare person from traditionally-trained finance that views portfolio selection as a multiple criteria problem, and it is a rare person from multiple criteria optimization that views portfolio selection as a single-criterion problem. Why would this be the case? Because there is both a *stochastic* and a *deterministic* side to portfolio selection. We now explain.

Consider a fixed sum of money to be fully invested in securities selected from a pool of n securities. Let there be a beginning of a holding period and an end of the holding period. Also, let x_i be the proportion of the fixed sum to be invested in the i-th security, $1 \le i \le n$. Here, the sum of the x_i equals one. Continuing, let r_i denote the random variable for the i-th security's return over the holding period. While the realized value of r_i is not known until the end of the holding period, all is not hopeless, however, as it is standardly assumed that all means μ_i, variances σ_{ii}, and covariances σ_{ij} of the r_i are known at the beginning of the holding period. Letting r_p denote the random variable for the return on the portfolio defined by the r_i and x_i over the holding period, we then have

$$r_p = \sum_{i=1}^{n} r_i x_i$$

Under the assumption that investors need only confine their interests to making money, the problem of standard portfolio selection is then to maximize the random variable r_p as in

$$\max\{r_p = \sum_{i=1}^{n} r_i x_i\}$$
$$s.t.\ \ \mathbf{x} \in S = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ \alpha_i \leq x_i \leq \omega_i \right\} \tag{1}$$

where S as given above is a typical feasible region (in *decision space*). While this may look like a linear programming problem, it is not a linear programming problem. Since the r_i are not known until the end of the holding period, but the x_i must be determined at the beginning of the holding period, what we have here is a *stochastic* programming problem. As defined in Caballero et al. (2001), if in a problem some parameters take unknown values at the time of making a decision, and these parameters are random variables, then the resulting problem is called a stochastic programming problem. Note that since S is deterministic, (1) is stochastic as a consequence of its objective being stochastic.

The difficulty with a stochastic programming problem is that its solution is not well defined. Hence, to solve (1) requires an interpretation. The approach taken in the literature (see for instance Stancu-Minasian, 1984; Prékopa, 1995; Slowinski and Teghem, 1990) is to ultimately transform the stochastic problem into an *equivalent deterministic problem* for solution. Equivalent deterministic problems typically involve the utilization of some statistical characteristic or characteristics of the random variables in question. For a problem with a stochastic objective such as (1), the equivalent deterministic problems enumerated in Caballero et al. (2001) result in the five *interpretations*

a) $\max\{E[r_p]\}$
   $s.t.\ \ \mathbf{x} \in S$
b) $\min\{Var[r_p]\}$
   $s.t.\ \ \mathbf{x} \in S$
c) $\max\{E[r_p]\}$
   $\min\{Var[r_p]\}$
   $s.t.\ \ \mathbf{x} \in S$
d) $\max\{P(r_p) \geq u\}$ for some chosen level of u
   $s.t.\ \ \mathbf{x} \in S$
e) $\max\{u\}$
   $s.t\ \ P(r_p \geq u) \geq \beta\}$ for some chosen level of β
   $\mathbf{x} \in S$

In case one is wondering how any of the above can be deterministic, recall that standard theory assumes that all means μ_i, variances σ_{ii}, and covariances σ_{ij} of the r_i are known at the beginning of the holding period. Even with this, how is one to know which is the most appropriate equivalent deterministic problem interpretation for a particular investor? At this point it becomes necessary to take a step back and delve into the rationale which leads from problem (1) to the interpretations (a) to (e).

Early seventeenth century mathematicians assumed that a gambler would be indifferent between receiving the uncertain outcome of a gamble or receiving in cash its expected value. In the context of portfolio selection, the gambler would be the investor, the gamble would be the portfolio return, and the *certain equivalent* would then be

$$CE = E[r_p]$$

Since an investor would obviously want to maximize the amount of cash received for certain, this rationale leads directly to interpretation (a). However, Bernoulli in 1738 discovered what has become known as the St. Petersburg paradox (because it was published in the *Commentaries from the Academy of Sciences of St. Petersburg*). A coin is tossed until it lands "heads". The gambler receives one ducat if it lands heads on the first throw, two ducats if it first lands "heads" on the second throw, four ducats if it first lands "heads" on the third throw, eight on the fourth, and so on (2^{h-1} ducats on the h-th throw). The expected value of the gamble is infinite, but in reality many gamblers would be willing to accept only a small number of ducats in exchange for the gamble. Hence, Bernoulli suggested not to compare cash outcomes, but to compare the "utilities" of cash outcomes. In the context of portfolio selection, it is assumed that the utility of a cash outcome is given by a utility function $U : \mathbb{R} \to \mathbb{R}$ (eventually an axiomatic foundation for the existence of a utility function was provided by Von Neumann and Morgenstern in 1947). Thus the certain equivalent (*CE*) becomes

$$U(CE) = E[U(r_p)]$$

That is, the utility of the certain equivalent should equal the expected utility of the uncertain portfolio return.

Since an investor would again want to maximize *CE*, and with *U* assumed to be increasing, this leads to the problem of Bernoulli's principle of maximum expected utility

$$\begin{aligned} &\max\{E[U(r_p)]\} \\ &s.t. \quad \mathbf{x} \in S \end{aligned} \tag{2}$$

That is, any x that solves (2) solves (1), and vice versa. Although Bernoulli's maximum expected utility problem is a deterministic equivalent to (1), we call it an equivalent "quasi" deterministic problem. This is because it is not fully *determined* in that it contains unknown utility function parameters and cannot be solved in its present form without further work. But since investors are assumed to be *risk averse*, i.e., the expected value $E[r_p]$ is always preferred over the uncertain outcome r_p, we at least know that utility function *U* is concave.

Two schools of thought have evolved for dealing with this difficulty. One involves trying to ascertain a particular investor's utility function and then using it in an attempt to directly solve (2). The other involves parameterizing *U* and then attempting to solve (2) for all possible values of its unknown parameters. Contemporary portfolio theory has evolved out of the second approach. Markowitz considered a parameterized *quadratic* utility function[1]

$$U(x) = x - (\lambda/2)x^2$$

Since *U* is normalized such that $U(0) = 0$ and $U'(0) = 1$, this leaves exactly one parameter λ, the *coefficient of risk aversion*. By this parameterization, Markowitz showed that precisely all potentially maximizing solutions of the equivalent "quasi" deterministic problem (2) for a risk averse investor can be obtained by solving interpretation (c)

$$\begin{aligned} &\max\{E[r_p]\} \\ &\min\{Var[r_p]\} \\ &s.t. \quad \mathbf{x} \in S \end{aligned}$$

for all **x**-vectors in *S* from which it is not possible to increase expected portfolio return without increasing portfolio variance, or decrease portfolio variance without decreasing expected portfolio return. As for terminology, all such x-vectors constitute the *efficient* set (in decision space) and the set of all images of the effi-

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[1] There is an anomaly with quadratic utility functions since they decrease from a certain point on. An alternative argument (not shown) instead of a quadratic utility function leading to the same result can be made by assuming that *U* is concave and increasing, and that $\mathbf{r}=(r_1, \ldots, r_n)$ follows a multinormal distribution.

cient points constitute the *nondominated* set in (variance, expected return) space. In finance it is common for authors to not only use the word "efficient" for qualifying vectors in decision space, but also for their images in (variance, expected return) space although some do take the time to say "mean-variance efficient". So as to minimize any confusion, we will adopt in this paper the operations research tradition of using "nondominated" to distinguish between the two.

Consequently, (c) is the ideal equivalent deterministic problem choice from among the five. In the extreme case of $\lambda = 0$ (risk neutrality) or $\lambda \to \infty$ (extreme risk aversion), we would obtain interpretations (a) or (b) as special cases of (c). Another extreme case of

$$U(x) = \begin{cases} 1, & x \geq u \\ 0, & x < u \end{cases}$$

with an unknown parameter u would yield interpretations (d) and (e). For instance, let u be the risk free interest rate. Then interpretation (d) would mean that the probability to receive at least the risk free interest rate is maximized. If $\mathbf{r} = (r_1, \ldots, r_n)$ follows a multinormal distribution, this is then equivalent to Roy's approach.

In this way, we have the stochastic and deterministic sides of portfolio selection. And we now see how there can be a legitimate disagreement on the number of criteria in standard portfolio selection. If only looking at the stochastic side, we would only see one objective, the single stochastic objective of portfolio return in (1). If only looking at the deterministic side, we would see typically see two objectives, the deterministic objectives of expected value and variance as a result of the choice of interpretation (c).

Although not a part of standard portfolio selection, let us briefly revisit the comment made earlier about skewness. Should the r_i be significantly skewed, a conceivable sixth equivalent deterministic problem interpretation could be

$$\begin{aligned} &\max\{E[r_p]\} \\ &\min\{Var[r_p]\} \\ &\max\{Skew[r_p]\} \\ &s.t. \quad \mathbf{x} \in S \end{aligned}$$

Although skewness shows up as a third criterion on the deterministic side, it is only there to help us better solve the single-criterion stochastic programming problem (1) on the stochastic side. This is why there can be debate about the criterion status of skewness. What you see depends upon which side you are standing.

3. With Multiple Objectives

We see three types of investors.

1. Those who truly are only interested in making money.
2. Those who have been intimidated into thinking that they must act as if they are only interested in making money, but actually have suppressed other concerns as well.
3. Those who have multiple objectives, are aware of the fact, and are in search of improved procedures.

There is little in this paper for investors of the first type. They are already well serviced by standard theory. While ideally suited for investors of the third type, the paper should also have a liberating effect on investors of the second type by giving them the theoretical support and confidence to proceed otherwise if they so desire.

With the previous section established, we now show how the theory of standard portfolio selection can be extended to include multiple stochastic and deterministic objectives. For the convenience of the rest of this section, let z_1 also be notation for denoting random variable r_p. Suppose that an investor's criterion situation is too complex to be covered by the tenets of standard theory. Then the investor would presumably be interested in maximizing some set of the z_i, with z_1 almost certainly included, selected from a list such as the following

1. $\max\{z_1 = \text{portfolio return}\}$
2. $\max\{z_2 = \text{dividends}\}$
3. $\max\{z_3 = \text{amount invested in R\&D}\}$
4. $\max\{z_4 = \text{social responsibility}\}$
5. $\max\{z_5 = \text{liquidity}\}$
6. $\max\{z_6 = \text{portfolio return over that of a benchmark}\}$
7. $\max\{z_7 = -\text{deviations from asset allocation percentages}\}$
8. $\max\{z_8 = -\text{number of securities in portfolio}\}$
9. $\max\{z_9 = -\text{turnover (i.e., costs of adjustment)}\}$
10. $\max\{z_{10} = -\text{maximum investment proportion weight}\}$
11. $\max\{z_{11} = -\text{amount of short selling}\}$
12. $\max\{z_{12} = -\text{number of securities sold short}\}$

The minus signs are used so that all objectives are in maximization form. On one hand, the first six objectives are stochastic as a result of the fact that they depend upon random variables associated with each of the n securities. On the other, the last six objectives are deterministic as they pertain to characteristics of the portfolio that would be known at the beginning of the holding period (i.e., as a result of the selection of the x_i). Whereas one person's set from the list might be $\{z_1, z_2, z_{10}\}$, another person's set might be $\{z_1, z_5, z_7, z_8, z_{11}\}$. If we let k be the number of selected objectives, in the case of the first investor k=3, and in the case of the second investor k=5. Of course, a standard investor's set would be $\{z_1\}$ in which case k=1.

As a guide to the rest of this section, consider Figure 1. At the top, as in Saaty's Analytic Hierarchy Process (Saaty, 1999), we have the investor's *overall focus*. In the box immediately below are the stochastic and deterministic objectives selected by the investor, say, from the above list. The notation here is that the number of selected stochastic objectives is η. Thus the number of selected deterministic objectives is $k-\eta$. As for the $D_{j_i}(\mathbf{x}), \eta+1 \le i \le k$, they are the functions of the deterministic objectives. For instance, if $D_{12}(\mathbf{x})$ were in the box, $D_{12}(\mathbf{x})$ would be a function that returns the number of securities with negative x_i weights.

In the third box we have

$$\begin{aligned} &\max\{E[U(z_{j1}, \ldots, z_{j\eta}, z_{j\eta+1}, \ldots, z_{jk})]\} \\ &s.t. \quad \mathbf{x} \in S \end{aligned} \tag{3}$$

which is a valid equivalent "quasi" deterministic problem provided that U is concave and increasing in each argument, and quadratic in each argument among $z_{j1}, \ldots, z_{j\eta}$. Assuming that all η stochastic objectives in the investor's set of k objectives are uncorrelated and that the coefficients of all stochastic objectives are from distributions whose means, variances and covariances are known at the beginning of the holding period, then each stochastic objective can be replaced in the equivalent "quasi" deterministic problem by a mean-variance pair in accordance with interpretation (c) of the previous section. The resulting equivalent deterministic problem is what is shown in the bottommost box. Note that all $k-\eta$ deterministic objectives indicated in an investor's $\{z_{j1}, \ldots, z_{j\eta}, z_{j\eta+1}, \ldots, z_{jk}\}$, being deterministic in the first place, are modeled unchanged in the equivalent deterministic problem.

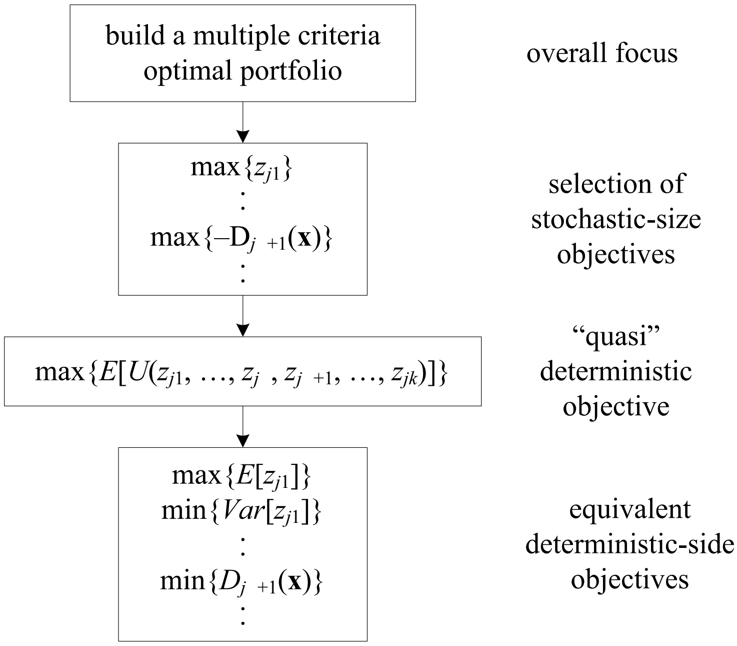

Figure 1. Hierarchical structure of the objectives and criteria in a multiple objective portfolio selection problem

As a practical matter, for stochastic objectives in which variation is small or not of noteworthy importance, it may be possible to select interpretation (a) instead of (c). This would simplify their representations in the equivalent deterministic problem. For example, suppose an investor's set is $\{z_1, z_5, z_8\}$. Here the investor's stochastic-side objectives are

$$\max\left\{ z_1 = \sum_{j=1}^{n} r_j x_j \right\}$$

$$\max\left\{ z_5 = \sum_{j=1}^{n} \ell_j x_j \right\}$$

$$\max\left\{ z_8 = -D_8(\mathbf{x}) \right\}$$

where ℓ_i is the random variable for the liquidity of the i-th security and D_8 is a function that returns the number of x_i weights that are nonzero. Using (c) for each stochastic objective, we have as objectives in the equivalent deterministic problem

$$\max\{E[z_1]\}$$
$$\min\{Var[z_1]\}$$
$$\max\{E[z_5]\}$$
$$\min\{Var[z_5]\}$$
$$\min\{D_8(\mathbf{x})\}$$

Now if we were to conclude that variation in portfolio liquidity is of much less importance than variation in portfolio return, then it may well be acceptable to use interpretation (a) instead of (c) for liquidity. Then this would result in the following equivalent deterministic problem

$$\begin{aligned} &\max\{E[z_1]\} \\ &\min\{Var[z_1]\} \\ &\max\{E[z_5]\} \\ &\min\{D_8(\mathbf{x})\} \end{aligned} \tag{4}$$

A major advantage of being able to use (a) instead of (c) is that a *Var* criterion is eliminated from the equivalent deterministic problem for each affected objective. Then, for each affected objective, it is only necessary to know the means of the individual security random variables at the beginning of the holding period, thus saving us from having to know any of their variances and covariances. In (4), we see that of the equivalent deterministic criteria resulting from the two originally stochastic objectives, only one is quadratic but two are linear.

4. Status of Standard Theory Computation

We now switch to matrix notation. When traditionally-trained finance people think of portfolio selection in the context of a mathematical programming problem, it is likely they would recollect the following "mean-variance" formulations

$$\begin{aligned} &\min\{\mathbf{x}^T \mathbf{\Sigma} \mathbf{x}\} \\ &s.t.\ \ \boldsymbol{\mu}^T \mathbf{x} \geq \rho \\ &\qquad \mathbf{x} \in S \end{aligned} \tag{P_ρ}$$

or

$$\begin{aligned} &\min\{\boldsymbol{\mu}^T \mathbf{x} - \lambda \mathbf{x}^T \mathbf{\Sigma} \mathbf{x}\} \\ &s.t.\ \ \mathbf{x} \in S \end{aligned} \tag{P_λ}$$

In these formulations, $\mathbf{\Sigma}$ is the covariance matrix of the individual security returns, ρ in (P_ρ) is a parameter to be ranged over various *target* values, and λ in (P_λ) is a parameter to be ranged over values $0 \to +\infty$ (please note that the λ of this section is different from the λ of Section 2). For solution, there are several approaches. With regard to (P_ρ), there are three. One is to repetitively optimize (P_ρ) employing different values of ρ each time using regular quadratic programming. Another is to use right-hand side parametric quadratic programming on the Kuhn-Tucker conditions. A third is to use Markowitz's variant of right-hand side parametric quadratic programming called the *critical line* algorithm (Markowitz, 1959). Whereas the first method is only able to achieve an approximation of the nondominated frontier, the last two methods enable the computation of the exact nondominated frontier.

With regard to (P_λ), we mention two approaches. One of course is to repetitively optimize (P_λ) employing different λ-values each time to obtain an approximated nondominated frontier. A second is to use objective-function parametric quadratic programming on the Kuhn-Tucker conditions to obtain the exact nondominated frontier. Unfortunately, right-hand side and objective-function parametric quadratic programming procedures are hard to come by and are not currently included in popular packages such as Matlab (Matlab, 2004), Cplex (Cplex, 2005), and LINGO (Schrage, 2004).

After inspection of (P_ρ) and (P_λ), multiple criteria optimization people would tend to read between the lines and view portfolio selection as the (bi-criterion) multiple criteria optimization problem

$$\begin{aligned} &\min\{\mathbf{x}^T \mathbf{\Sigma} \mathbf{x}\} \\ &\max\{\boldsymbol{\mu}^T \mathbf{x}\} \\ &s.t.\ \ \mathbf{x} \in S \end{aligned} \tag{5}$$

Why would a multiple criteria person recognize (P_ρ) and (P_λ) as (5)? Because the person would know about the *e*-constraint method of multiple criteria optimization (for example, described in Steuer, 1986; Chap. 8). In the *e*-constraint method, all objectives but one are converted to inequality constraints ($\geq$ for "max" objectives, $\leq$ for "min" objectives) with the e_i right-hand sides set to target values at the instinct of the user. However, methods used for solving (5) typically involve the same methods as for (P_ρ) and (P_λ). Actually, this is not surprising as multiple criteria optimization is usually not concerned with bi-criterion mathematical programming problems since they can normally be solved using techniques from the single-criterion literature. Rather, multiple criteria optimization's real interest is in problems with three or more objectives where surfaces have to be computed and searched, i.e., problems on which extensions of single-criterion methods are of little help.

5. Multiple Criteria Nondominated Sets

The nondominated sets of standard mean-variance equivalent deterministic problems are no problem. We know they are piecewise *parabolic* in (variance, expected return) space, and thus are piecewise *hyperbolic* in (standard deviation, expected return) space. Even though theory and computation are typically carried out in terms of variance, being piecewise *hyperbolic* in (standard deviation, expected return) space is relevant because nondominated frontiers are typically presented to users with standard deviation being on the horizontal axis. Part, if not all, of the reason for this is that variance is measured in strange units, (%/time period) squared. Standard deviation on the other hand is measured in percent per time period, a much more workable unit. Percent per time period has the added convenience of being the same unit in which expected return is measured.

Consider Figure 2. With standard deviation on the horizontal axis, Figure 2 *right* is drawn to represent a nondominated frontier. Portrayed with five piecewise hyperbolic segments, such would be typical of a problem with not more than about 10-15 securities. Along a given hyperbolic segment, the securities in a portfolio remain the same. Only their proportions change as we move from point to point. In Figure 2 *left*, the piecewise linear path is drawn to represent the set of all inverse images (in decision space) of all points on the nondominated frontier. In particular, the *inverse image set* of each individual hyperbolic segment is an individual (straight) line segment. For example, the line segment $\mathbf{t}^3$ to $\mathbf{t}^2$ inclusive is the inverse image set for the hyperbolic segment $\mathbf{z}^{t3}$ to $\mathbf{z}^{t2}$ inclusive. Since the five hyperbolic segments comprise the nondominated set, the piecewise linear path is the efficient set. Intermediate points along the path such as $\mathbf{t}^1$, $\mathbf{t}^2$, $\mathbf{t}^3$ and $\mathbf{t}^4$ are called *turning points*. It is only at these points that new securities can enter, and old securities can leave, a nondominated portfolio.

Suppose an equivalent deterministic problem has one quadratic and two or more linear criteria. We will call such problems *extended EDP*s (where EDP stands for equivalent deterministic problem) to distinguish them from the *mean-variance EDP*s of standard theory. Other than for the theoretical work of Guddat (1976), we are unaware of anyone else who has looked into the structure of the efficient and nondominated sets of problems such as extended EDPs. However, from the working paper of Hirschberger et al. (2005b), we are able to share some preliminary findings. Whereas a mean-variance EDP has a nondominated *frontier*, an extended EDP has a nondominated *surface*. Whereas a mean-variance EDP is piecewise hyperbolic in (standard deviation, expected return) space, the nondominated surface of an extended EDP is platelet-wise *hyperboloidic* in (standard deviation, expected return, extra linear objectives) space. And whereas the inverse image set of the nondominated frontier of a mean-variance EDP is a piecewise linear path in $S \in \mathbb{R}^n$, the inverse image set of the nondominated surface of an extended EDP is a connected union of polyhedra in $S \in \mathbb{R}^n$. Even though S is normally of dimensionality $n-1$, it is believed that the dimensionality of a given polyhedron in a connected union will never exceed the number of linear objectives. This however has not yet been proved.

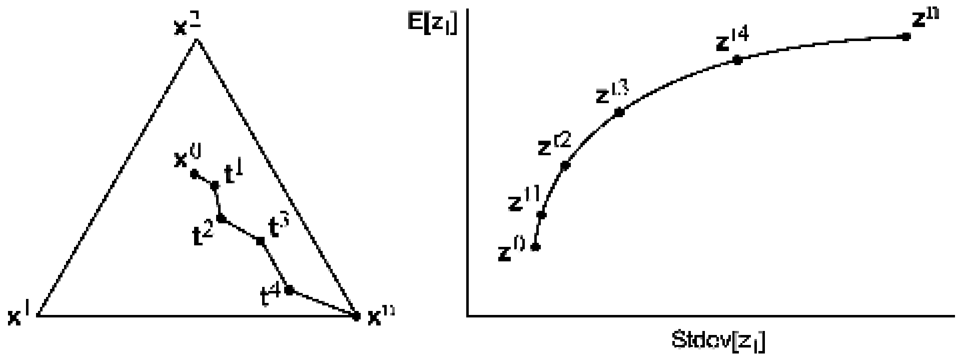

Figure 2. Efficient and nondominated set renditions of a standard theory mean-variance equivalent deterministic problem

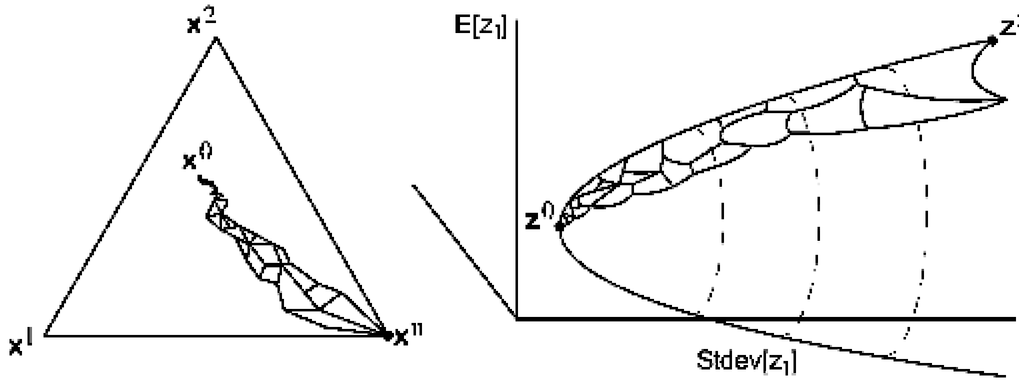

Figure 3. Efficient and nondominated set renditions of an equivalent deterministic problem with one quadratic and two linear criteria

To illustrate an extended EDP that has one quadratic and two linear criteria, Figure 3 *right* is drawn to portray a nondominated surface and the platelets (like on the back of a turtle) of which it is composed. Note how the sizes of the platelets generally decrease as we move down the surface from $\mathbf{z}^n$ to $\mathbf{z}^0$. This is normal. Also, it is normal for all platelets to not always have the same number of what we call *platelet corner points*. Figure 3 *left* is drawn to represent a connected union of inverse image sets that could be the efficient set corresponding to the nondominated surface. Instead of being 1-dimensional as in mean-variance EDPs, here the individual inverse image sets of the individual hyperboloidic platelets are 2-dimensional polyhedra. Note how the inverse image set extreme points and the platelet corner points, even though their densities increase as we approach $\mathbf{x}^0$ and $\mathbf{z}^0$, could function as discretized representations of the efficient and nondominated sets, respectively.

6. Permissibility of Dense Covariance Matrices

With regard to computation, it is interesting to look at the history of portfolio selection. When Markowitz published his paper (Markowitz, 1952) on portfolio selection in 1952 and his algorithm in 1956 (Markowitz, 1956), computers really were not up to the task of computing a nondominated frontier. For a handful of securities, yes, but not for the asset universes of several hundred securities awaiting portfolio selection on Wall Street. There were two obstacles. One was CPU time. The other was *core storage* (mainframe counterpart of what is known today as RAM). CPU time was the lesser obstacle of the two because it could always be gotten around by waiting longer, but core storage was a "hard" constraint. There was never enough of it, and once you hit the maximum, there was usually nothing you could do but to re-program your code to use less space. In fact, up until the early 1970s, if a researcher wished to run a job requiring 256KB of core storage at a major university, it would not have been out of the question to have to get permission from the computing center director first.

In portfolio selection, both obstacles came into play principally because of dense $n \times n$ covariance matrices. During the early 1980s, to give larger problems a better chance to run, some very innovative techniques were developed by Markowitz and Perold (1981a,b), Perold (1984), and others. To avoid having to confront dense covariance matrices, the approaches focused on methods for diagonalizing the covariance matrix structure. These methods are very clever. They enable one to go directly from historical data to a diagonalized covariance matrix structure by adding only a few extra variables and constraints to the formulation. With diagonalized covariance matrix structures, not only are core storage or RAM requirements reduced, but there are substantial savings in the number of arithmetic operations required for the computation of a nondominated frontier.

The downside of the above methods is that one can generally figure that there is a loss of information. Thus there is likely to be a difference between the frontier obtained and the one desired. Nevertheless, the approaches became popular because they extended the reach of portfolio selection to problems with up to 500 securities by the end of the 1980s, and beyond 500 securities in the 1990s. Konno and Suzuki (1992) proposed yet another diagonalization strategy. It is interesting because it does not involve a loss of information. Being based upon Cholesky factorization, it requires that a covariance matrix be initially in hand and that it be invertible. While this method, when applicable, will produce a true nondominated frontier, the difficulty is that most large covariance matrices are not invertible. Regardless of which method one prefers to use, diagonalization has essentially become a staple of large-scale optimization to the extent, without thinking, it is almost always thought to be a mandatory first step, but this no longer need be the case.

With the speed of computers today, and the virtual disappearance of any kind of storage or memory as a constraint, the need to always diagonalize the covariance structure, unless needed for some other purpose, is no longer a requirement to bring large-scale portfolio optimization into the realm of computational feasibility. We are entering a new era. Dense covariance matrices of up to 2000-3000 variables no longer pose the problems they once did. According to Hirschberger et al. (2005c), the computation of exact nondominated frontiers of large problems with dense covariance matrices is now possible in reasonable time even on a laptop. For instance, on a 1.6GHz Centrino laptop, we are now experiencing 10-15 minute times to compute the exact nondominated frontier of a standard EDP with a 100% dense covariance matrix and n=1000, and 40-50 minute times to compute the exact nondominated frontier of a standard EDP with a 100% dense covariance matrix and n=1500.

7. Future Directions

Because we are in the early stages of multiple criteria portfolio optimization, often when research is conducted, more questions may be raised than answered. In this regard, we see many research projects ahead, seven of which are outlined below.

1. A computational study of the numerical characteristics of the nondominated frontiers of different types of mean-variance portfolio selection problems and of the time to compute the nondominated frontiers on different software platforms.
2. A computational study on the loss of information resulting from different schemes for diagonalizing the covariance matrix structure in mean-variance portfolio optimization.
3. Development of algorithms for computing the efficient and nondominated sets of extended EDPs with 3 or more linear criteria.

4. Using evolutionary algorithms to obtain discretized representations of the nondominated sets of extended EDPs with non-smooth characteristics. . ctrg
   a) Cardinality constraints
   b) Semi-continuous variables
5. A computational comparison of different methods for interactively searching the nondominated sets of extended EDPs and an evaluation of the decision-making style of each method.
   a) Projected line search procedures (Korhonen and Wallenius, 1988; Korhonen and Karaivanova, 1989).
   b) Dispersed sampling procedures (Steuer, 1986).
   c) Classification procedures (Meittinen, 1999).
   d) Interactive decision maps (Lotov et al., 2004).
6. How to best handle the extra linear variables that are necessary when modeling.
   a) How to best handle the extra linear variables that are necessary when modeling
   b) "Exception-to-the-rule" constraints
   c) Most of the deterministic objectives
7. A study of the advantages and disadvantages of using mean absolute deviation (MAD) in place of variance in portfolio selection.

Concerning (1), we know of no papers of contemporary value that report on the time it takes to solve for the nondominated frontiers of problems of different sizes. We speculate that the reason for this is that there has been no way to easily generate test problem covariance matrices other than from real data. However, the random generation of covariance matrices is now possible using the routine of Hirschberger et al. (2005a), so the roadblock to such a study is now removed.

Also using this routine along with the ability to solve for the exact nondominated frontiers of problems with large dense covariance matrices, we are now for the first time in a position to conduct a study such as described in (2) about the amount of information lost in large problems when the covariance matrix structure is diagonalized.

With research underway for solving for the nondominated surface of an extended EDP with one quadratic and two linear objectives, the next step is to extend the algorithm so that it can deal with problems with one quadratic and three or more linear objectives as set out in (3).

Item (4) deals with the effectiveness of evolutionary algorithms (see Deb, 2001) for being able to develop usable discretized representations of the nondominated set in problems that are made non-smooth, for instance, by the conditions listed.

Item (5) deals with the evaluation of different methods for searching the nondominated set using procedures whose capabilities and feel offer an opportunity to be consistent with the decision-making style requirements of, say, a portfolio manager.

When modeling many of the deterministic objectives listed in Section 3, extra linear variables are generated in the formulation usually at the rate of one per security in the asset universe. Study (6) is directed at how to best handle algorithmically the increase in problem size that results.

With regard to (7), some, for instance Mansini et al. (2003), have proposed the use of MAD in place of variance. The advantage of this is that MAD can be modeled linearly. Since this is likely to result in a loss of information, too, the pros and cons of this proposition need to be carefully studied.

As seen throughout this paper, researching multiple objectives in portfolio selection is challenging. However, it is worthwhile because as soon as enough results can be accumulated, portfolio selection will then no longer be a one-size-fits-all affair. Portfolio selection will then have the ability to adapt itself to meet the criterion needs of most any investor, and in this way, become a much more robust tool for financial analysis.

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