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Computational experience concerning payoff tables and minimum criterion values over the efficient set

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Abstract: Minimum criterion values from payoff tables have often been used in multiple objective linear programming (MOLP). The assumption has often been that the minimum criterion values from payoff tables provide reasonably accurate estimates of the minimum criterion values over the efficient set. In this paper, however, we report computational experience that demonstrates that the discrepancies between the payoff table minimums and the minimums over the efficient set can often be large. This tends to imply that the field of multiple objective programming needs a better method than payoff tables for estimating the minimum criterion values over the efficient set. The paper concludes with a discussion of a simplex-based procedure for deterministically computing the minimum criterion values over the efficient set that has potential in large MOLP applications.

Keywords: Multiple criteria programming

1. Introduction and terminology

Minimum criterion values over the efficient set are of interest in multiple objective programming in order to characterize the ranges of the criterion values over the efficient set. The process of using *payoff tables* (defined shortly) to obtain estimates of the minimum criterion values over the efficient set has been integrated into a number of interactive multiple objective linear programming procedures [1,2,9,10,13,15,16, and 18]. Also, in order to allow a decision maker to *size* his or her multiple objective problem [17], knowledge of the minimum criterion values is required when attempting to graphically display the criterion value ranges over the efficient set.

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As pointed out by Weistroffer [21] and Dessouky, Ghiassi, and Davis [4], payoff tables only provide *estimates* of the minimum criterion values over the efficient set. The purpose of this paper is to report computational experience concerning the degree to which payoff tables might furnish good or bad estimates of the minimum criterion values over the efficient set.

To establish notation and terminology, consider the multiple objective linear program (MOLP)

$$\begin{array}{ll} \max & \{c^1x = z_1(x)\}, \\ \max & \{c^2x = z_2(x)\}, \\ & \vdots \\ \max & \{c^kx = z_k(x)\}, \\ \text{s.t.} & x \in S = \{x \in \mathbb{R}^n \mid Ax \leqslant b,\ x \geqslant 0,\ b \in \mathbb{R}^m\}, \end{array}$$

as discussed in [5,6,7,11,15] and others. Alternately, the above MOLP can be written in *vector*-

maximum form as

$$\text{'max'}\{z(x) = Cx \mid x \in S\}$$

where $z(x) = (z_1(x), \ldots, z_k(x))$, C is the $k \times n$ *criterion matrix* whose rows are the c^i, and 'max' means that all *efficient* solutions are to be found. A point $\bar{x} \in S$ is efficient if and only if there does not exist an $x \in S$ such that $z(x) \geqslant z(\bar{x})$, $z(x) \neq z(\bar{x})$. The set of all efficient solutions is denoted E. Let $\bar{x} \in S$, then, if $\bar{x} \in E$, $z(\bar{x})$ is a *nondominated* criterion vector, and if $\bar{x} \notin E$, $z(\bar{x})$ is a *dominated* criterion vector.

Let $z_i^{\max}$ and z_i^{emin} denote the maximum and minimum values of the i-th objective over the efficient set, respectively. The $z_i^{\max}$ are easy to obtain because

$$z_i^{\max} = \max\{z_i(x) = c^i x \mid x \in S\}.$$

We simply maximize each of the objectives individually over the feasible region S. The z_i^{emin} are not as easy to obtain because

$$z_i^{\text{emin}} = \min\{z_i(x) = c^i x \mid x \in E\}.$$

The difficulties are that, in general, E is not known explicitly, and in all but the most trivial cases, E is nonconvex. Dessouky, Ghiassi, and Davis [4] discuss three *heuristics* for solving for the z_i^{emin}. In this paper, we discuss three *deterministic* methods for computing the z_i^{emin}, concentrating primarily on the third method which is a simplex-based procedure that can be used on MOLPs of any size.

Let $x_{\max}^i$ denote the solution resulting from the i-th individual maximization over S:

$$\max\{c^i x \mid x \in S\}.$$

Then, using the $x_{\max}^i$, a *payoff table* is constructed as in Table 1.

In Table 1, the i-th row is the criterion vector $z(x_{\max}^i)$ corresponding to the solution obtained from the i-th individual maximization. Also, since $z_i^{\max} = c^i x_{\max}^i$, we observe that the $z_i^{\max}$ are found along the main diagonal of the payoff table.

Table 1
Payoff table

	z_1	z_2		z_k
$z(x_{\max}^1)$	$c^1 x_{\max}^1$	$c^2 x_{\max}^1$	$\cdots$	$c^k x_{\max}^1$
$z(x_{\max}^2)$	$c^1 x_{\max}^2$	$c^2 x_{\max}^2$		$c^k x_{\max}^2$
	$\vdots$			
$z(x_{\max}^k)$	$c^1 x_{\max}^k$	$c^2 x_{\max}^k$		$c^k x_{\max}^k$

Using a payoff table, a popular way to estimate the z_i^{emin} is to scan the columns of the payoff table to determine the quantities

$$z_i^{\text{pmin}} = \min_{1 \leqslant j \leqslant k} \{c^i x_{\max}^j\}, \quad i = 1, \ldots, k.$$

We call the z_i^{pmin} the *payoff table column minimum* values. Then, the z_i^{pmin} are used as estimates of the z_i^{emin}. Unfortunately, as illustrated by the numerical example of Section 2 and in the computational experience of Section 3, the z_i^{pmin} are often such poor estimates of the z_i^{emin} as to call into question the whole process of using payoff tables to estimate the z_i^{emin}.

2. Numerical example

Consider the MOLP numerical example of Table 2.

As determined by ADBASE [19], the MOLP has 25 efficient extreme points. In the individual maximizations of the objectives, it turned out that, because of alternative optima, $x_{\max}^2$ and $x_{\max}^3$ were inefficient. Thus, in the resulting payoff table of Table 3, the second and third rows are formed by dominated criterion vectors. Also given in Table 3 are the $z_i^{\max}$, z_i^{pmin}, and z_i^{emin}. The first arrow at the bottom of Table 3 means that $z_1^{\text{emin}} > z_1^{\text{pmin}}$. The second arrow indicates that $z_2^{\text{emin}} < z_2^{\text{pmin}}$ and that 5 of the 25 efficient extreme points had z_2 values less than z_2^{pmin}. The third arrow means that $z_3^{\text{emin}} < z_3^{\text{pmin}}$ and that only one efficient extreme point had its z_3 value less than its payoff table column minimum. We have a similar situation with the fourth arrow. Overall, 6 of the 25 efficient extreme points were in violation of one or more payoff table column minimums.

One might speculate that the difficulty with payoff tables would be less if they were constructed using only nondominated criterion vectors. To ensure that each maximizing criterion vector is nondominated, let us *lexicographically* maximize each of the objectives in (1, 2, 3, 4), (2, 3, 4, 1), (3, 4, 1, 2), and (4, 1, 2, 3) order, respectively. For instance, (4, 1, 2, 3) means that we solve *lex max*$\{c^4 x, c^1 x, c^2 x, c^3 x \mid x \in S\}$. Doing the lexicographic maximizations, we obtain Table 4.

Table 2
MOLP numerical example

	x_1	x_2	x_3	x_4	x_5	x_6	x_7		
Objectives	−2	1	2	−1	1	2	−1	max	
	−1	−2		−2	3	1		max	
	2			−2		−2	−2	max	
	2	−1	1	1			3	max	
s.t.	1	1	3		3	2		$\leqslant$	61
		3	2	4				$\leqslant$	72
	5	3			5	4	4	$\leqslant$	76
	4	2		4		4		$\leqslant$	51
	5	2		3	1	4		$\leqslant$	66
	2	2		4	4	4	5	$\leqslant$	59
	3		2		5	1	2	$\leqslant$	77

all $x_i \geqslant 0$

Table 3
Payoff table information generated from individual maximizations

	z_1	z_2	z_3	z_4	
$z(x^1_{\max})$	49.17	12.75	−25.50	11.83	nondominated
$z(x^2_{\max})$	14.75	44.25	0.00	0.00	dominated
$z(x^3_{\max})$	−25.50	−12.75	25.50	25.50	dominated
$z(x^4_{\max})$	10.04	−8.30	−0.34	59.60	nondominated
$z^{\max}$	49.17	44.25	25.50	59.60	
z^{pmin}	−25.50	−12.75	−25.50	0.00	
z^{emin}	3.50	−35.15	−28.70	−4.27	
	↑	↓5	↓1	↓1	

Table 4
Payoff table information from lexicographic maximizations

	z_1	z_2	z_3	z_4	
$z(x^{1,2,3,4}_{\max})$	49.17	12.75	−25.50	11.83	nondominated
$z(x^{2,3,4,1}_{\max})$	18.00	44.25	0.00	1.62	nondominated
$z(x^{3,4,1,2}_{\max})$	6.67	−12.75	25.50	41.58	nondominated
$z(x^{4,1,2,3}_{\max})$	10.04	−8.30	−0.34	59.60	nondominated
$z^{\max}$	49.17	44.25	25.50	59.60	
z^{pmin}	6.67	−12.75	−25.50	1.62	
z^{emin}	3.50	−35.15	−28.70	−4.27	
	↓7	↓5	↓1	↓1	

In this case, our payoff table difficulties have gotten worse. Now a total of 12 out of the 25 efficient extreme points violate one or more of the payoff table minimums. Thus, it is hard to argue that the occurrence of dominated criterion vectors is the cause of the problem. (Note that when lexicographically maximizing the objectives, a z_i^{pmin} cannot be less than its corresponding z_i^{emin} as occurred in the first column of Table 3.)

3. Computational experience

The MOLP of the previous section is not atypical. Such difficulties are widespread. To understand how widespread the difficulties are with regard to using the z_i^{pmin} to estimate the z_i^{emin}, computational experiments were conducted with randomly generated problems. The experiments were conducted to determine if there were any effects due to the size of the criterion cone (generated by the c^i), the number of objectives, or problem size (number of constraints × number of objectives). In each experiment the sample size was 10. That is, 10 MOLPs were randomly generated and solved. In these MOLPs, the right-hand side elements were randomly drawn from the interval of integers [50,100]. After first providing for a 25% zero-density in the A-matrix, the remaining A-matrix elements were drawn from the interval of integers $[-1,20]$. In the experiments, we generated three types of criterion cones: narrow, intermediate, and wide-open. In the *narrow* criterion cone case, the C-matrix elements were drawn from the interval of integers [0,20], in the *intermediate* criterion cone case, they were drawn from the interval $[-10,20]$, and in the *wide-open* criterion cone case, they were drawn from the interval $[-20,20]$.

Table 5 shows one of the experiments in detail. The contents of Table 5, in which $5 \times 10 \times 10$ means 5 objectives, 10 constraints and 10 variables, are now explained. The first problem of the experiment, Problem 1, had 87 efficient extreme points. With regard to the first objective, the *hidden percentage* is 26.60%. By this we mean the percentage of the criterion value range over the efficient set that is 'hidden' below the payoff table column minimum. That is,

$$\frac{z_1^{\text{pmin}} - z_1^{\text{emin}}}{z_1^{\max} - z_1^{\text{emin}}}(100) = 26.60.$$

Also, it is found that 3 of Problem 1's efficient extreme points have z_1 values less than z_1^{pmin}. As seen, the maximum hidden percentage for Problem 1 is 47.54%. The dashed entries in Table 5 mean that the z_i^{pmin} in question correctly estimated their z_i^{emin}. In the last column of Table 5 for Problem 1, the average percentage of the criterion value ranges (over E) below the z_i^{pmin} is 21.33%. That is,

$$\frac{1}{5}\sum_{i=1}^{5}\frac{z_i^{\text{pmin}} - z_i^{\text{emin}}}{z_i^{\max} - z_i^{\text{emin}}}(100) = 21.33.$$

In total, 10 of Problem 1's 87 efficient extreme points had one or more z_i values less than their corresponding z_i^{pmin}.

Tables 6, 7 and 8 give the results of experiments in which we controlled for the size of the criterion cone, the number of objectives, and problem size. For each experiment, the tables report:

(1) the average hidden percentage,

(2) average maximum hidden percentage per problem,

(3) the average percentage of the total number

Table 5
$5 \times 10 \times 10$ experiment with $[-20,20]$ criterion cone

Problem	Efficient extreme points	Hidden percentages and number of efficient extreme points below					
1	87	26.60	47.54	12.84	–	19.64	21.33
		3	5	1	–	2	10
2	109	2.97	0.65	–	1.52	33.22	7.17
		1	1	–	1	3	5
3	117	–	4.37	20.96	16.38	11.58	10.66
		–	2	1	1	2	5
4	190	13.69	10.65	11.13	15.61	–	10.22
		6	2	3	3	–	14
5	92	–	38.49	–	–	22.16	12.13
		–	1	–	–	1	1
6	104	28.16	3.58	12.18	19.40	20.70	17.40
		7	1	2	7	2	15
7	108	–	–	–	1.76	9.94	2.34
		–	–	–	1	4	5
8	201	8.98	–	32.53	21.92	2.85	13.26
		3	–	2	5	1	11
9	182	36.40	4.90	14.90	25.61	0.57	16.48
		3	3	6	5	2	19
10	44	1.81	51.06	–	–	43.00	19.17
		1	2	–	–	2	3

Table 6
Controlling for size of criterion cone

Averages per problem	5×10×10	5×10×10	5×10×10
	[0,20] cone	[−10,20] cone	[−20,20] cone
(1) Hidden %-age	8.47%	15.55%	13.06%
(2) Maximum hidden %-age	23.36%	34.56%	31.39%
(3) % of ranges in violation	56.00%	74.00%	74.00%
(4) No. of eff. ext. pts.	53.7	104.5	123.4
(5) No. of eff. ext. pts. below	5.9	9.2	8.8
(6) % of eff. ext. pts. below	11.75%	8.99%	7.11%

Table 7
Controlling for number of objectives

Averages per problem	[−10,20] criterion cone		
	3×10×10	5×10×10	7×10×10
(1) Hidden %-age	3.94%	15.55%	12.33%
(2) Maximum hidden %-age	9.29%	34.56%	33.27%
(3) % of ranges in violation	33.33%	74.00%	82.86%
(4) No. of eff. ext. pts.	19.0	104.5	222.9
(5) No. of eff. ext. pts. below	1.4	9.2	16.1
(6) % of eff. ext. pts. below	9.17%	8.99%	7.95%

Table 8
Controlling for problem size

Averages per problem	[−10,20] criterion cone		
	4×8×8	4×16×16	4×24×24
(1) Hidden %-age	8.43%	15.50%	14.17%
(2) Maximum hidden %-age	17.72%	30.56%	28.56%
(3) % of ranges in violation	47.50%	92.50%	87.50%
(4) Efficient extreme points	21.7	294.3	881.1
(5) Efficient ext. pts. below	4.0	14.7	18.7
(6) % of eff. ext. pts. below	18.42%	5.12%	2.48%

of the criterion value ranges per problem that are incorrectly specified by $[z_i^{\mathrm{pmin}}, z_i^{\max}]$,

(4) the average number of efficient extreme points per problem,

(5) the average number of efficient extreme points per problem that have one or more z_i values below their corresponding z_i^{pmin}, and

(6) the average percentage of the total number of efficient extreme points per problem that have one or more z_i values below their corresponding z_i^{pmin}.

From Tables 6, 7 and 8 we see that the likelihood of a given z_i^{emin} being less than its corresponding z_i^{pmin} can easily be in the 70 to 90% range in problems with more than 100 efficient extreme points. Consequently, the difficulties do not appear to go away as problem size increases. Also, we see that many, if not most, problems tend to have at least one range whose hidden percentage is 30% or more. Thus, extreme caution is advised when using payoff tables to estimate the ranges of the criterion values over the efficient set, because, in many cases, the z_i^{pmin} may not even be in the same ballpark as the z_i^{emin}. The only partially encouraging note is that the percentage of efficient extreme points that have one or more z_i values below their corresponding z_i^{pmin} tends to decrease with the total number of efficient extreme points. All of the experiments were run using ADBASE [19].

4. Simplex-based approach

The results of the previous section make it clear that the field of multiple objective linear programming needs a better method than payoff tables to compute minimum criterion values over the efficient set.

In [4], three heuristics were presented for computing the minimum criterion values z_i^{emin}. In this paper, we now look at three deterministic approaches. The first is to use a vector-maximum code such as ADBASE [19] or EFFACET [12] to compute all efficient extreme points. Perhaps, such vector-maximum codes could be augmented with a pre-processing routine as suggested by Gal [8] to eliminate *deletable* objectives (objectives whose gradients are nonnegative linear combinations of other objective gradients), should any exist. That would boost these codes to maximum speed for the generation of all efficient extreme points. Then, by examining the components of the criterion vectors of each of the efficient extreme points, the z_i^{emin} are determined. Even if running at maximum speed, the amount of computer time required for the vector-maximum generation of all efficient extreme points would still be too large for this approach to be a serious candidate except with relatively small MOLPs.

Another approach is to solve the following *primal–dual feasible program*

$$
\begin{aligned}
\min \quad & \{ z_i(x) = c^i x \}, \\
\text{s.t.} \quad & Ax \leqslant b, \\
& x \geqslant 0, \\
& A^{\mathrm{T}} u - C^{\mathrm{T}} \lambda \geqslant 0, \\
& \lambda \geqslant e, \\
& u^{\mathrm{T}} b - \lambda^{\mathrm{T}} C x = 0, \\
& u \geqslant 0,
\end{aligned}
$$

where $e \in \mathbb{R}^k$ is the sum vector of ones. Justification for this approach is derived from Kornbluth [14] from which we have the result that $\bar{x} \in E$ if and only if there exists a $\bar{u} \in \mathbb{R}^m$ and a $\bar{\lambda} \mathbb{R}^k$ such that $(\bar{x}, \bar{u}, \bar{\lambda})$ solves the primal–dual feasible program.

The difficulty with this approach is the size of the primal–dual feasible program. It has $n + k + 1$ more constraints and $m + k$ more variables than the original MOLP. This results in roughly twice as many rows and twice as many columns. Also, the last constraint of the formulation is highly nonlinear involving each λ-variable in at least one nonlinear term. Consequently, this approach is out of the question on large MOLPs.

The third approach is a simplex-based procedure that is based upon the well-known result that each efficient extreme point is connected to every other efficient extreme point by means of a path of efficient edges [3]. Thus, every hyperplane $\{ x \in \mathbb{R}^n \mid c^i x = \bar{z} \}$, where $\bar{z} \in [z_i^{\text{emin}}, z_i^{\max}]$, intersects at least one efficient edge of S. With this observation, we have the following reduced feasible region algorithm for computing the i-th criterion value minimum over E.

Step 1. Let $\bar{x}$ designate an efficient extreme point of S. (We can either let $\bar{x}$ be the extreme point associated with z_i^{pmin} from a lexicographically generated payoff table, or use one of the methods discussed in Steuer [20, Section 9.6] to generate an $\bar{x}$.)

Step 2. Let reduced feasible region $\bar{S} = S$.

Step 3. Does there exist an edge of $\bar{S}$ emanating from $\bar{x}$ that is efficient and has a negative $c_j - z_j$ reduced cost value with respect to c^i? If yes, go to Step 4. If no, go to Step 5.

Step 4. Pivot along the edge to the adjacent extreme point. Let $\bar{x}$ designate the new extreme point. Go to Step 3.

Step 5. Let reduced feasible region $\bar{S} = \bar{S} \cap \{ x \in \mathbb{R}^n \mid c^i x \leqslant c^{\bar{x}} \}$.

Step 6. Does there exist an extreme point of $\bar{S}$ on the $\{ x \in \mathbb{R}^n \mid c^i x = c^i \bar{x} \}$ hyperplane from which there exists an edge that is efficient and has a negative $c_j - z_j$ reduced cost value with respect to c^i? If yes, go to Step 4. If no, go to Step 7.

Step 7. Stop with $z_i^{\text{emin}} = c^i \bar{x}$.

Although not especially economical, the simplex-based procedure appears to be the only deterministic approach yet proposed with any degree of practicality in large scale MOLP applications.

5. Illustration of simplex-based algorithm

Recall the MOLP numerical example of Section 2 that has 25 efficient extreme points. As determined by EFFACET [12], the MOLP has 8 maximally efficient facets as defined in Table 9.

Figure 1 shows how the 25 efficient extreme points are connected by 45 efficient edges. In Figure 1 the z_1 value of each efficient extreme point is written next to its node.

Applying the algorithm of Section 4 to determine z_1^{emin}, let us start at the efficient extreme point that generated z_1^{pmin} in the payoff table of Table 4. Thus we start at x^{18} whose $z_1 = 6.67$. Let us pursue the $\langle x^{18}, x^{23} \rangle$ edge in the 3rd efficient facet to x^{23} whose $z_1 = 3.60$. We find that x^{23} is a local minimum for z_1 over E. We then determine that the $c^1 x = 3.60$ hyperplane intersects, for instance, the relative interior of the $\langle x^{14}, x^{13} \rangle$ efficient edge which is in the 1st efficient facet. We

Table 9
Efficient extreme points associated with the different maximally efficient facets

Facet	Indices of efficient extreme points
1	1, 2, 3, 5, 6, 7, 8, 11, 13, 14, 16, 17
2	1, 3, 4, 5
3	3, 5, 8, 9, 11, 12, 16, 17, 18, 23
4	3, 4, 5, 9, 10, 12, 20, 21
5	7, 15, 16
6	9, 18, 19, 24
7	9, 19, 20, 25
8	15, 16, 22, 23

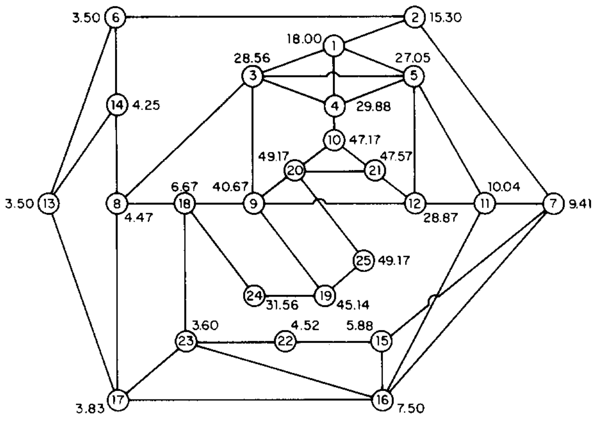

Figure 1. Graph of efficient extreme points, efficient edges, and z_1 values

then follow the $\langle x^{14}, x^{13} \rangle$ edge to x^{13} whose $z_1 = 3.50$. We find that x^{13} is a local minimum over E. Since the $c^1x = 3.50$ hyperplane does not intersect any efficient edge leading to lower ground, we terminate with x^{13} as a global minimum for z_1 over E. Thus $z_1^{\text{emin}} = 3.50$.

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