Goal programming with linear fractional criteria

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In this paper we present an extension of goal programming to include linear fractional criteria. The extension forms a natural link between goal programming (GP) and multiple objective linear fractional programming (MOLFP).

1. Introduction

During the 1970's, two major solution approaches emerged for solving multiple criterion 'linear-programming-type' problems. They are goal programming (GP) and multiple objective linear programming (MOLP).

In goal programming (Charnes and Cooper [2], Lee [16], Ignizio [8], and Kornbluth [11]), we find the lexicographic minimum of

$$\bar{P}=\{P_1,P_2,\ldots,P_r,\ldots,P_R\},$$
$$\text{s.t. } c^ix+d_i^- - d_i^+ = z_i^* \quad \text{for } i=1,2,\ldots,k,$$
$$x\in S,$$
$$d^-,d^+\geqslant 0\in \mathbf{R}^k,$$

where:

(a) P_r represents the *deviational variable objective function* of the rth *priority level* (where the deviational variable objective function is a weighted-sum of the deviational variables associated with that priority class);

(b) the c^ix are the (linear) criterion functions;

(c) the d_i^- and d_i^+ are the respective underachievement and overachievement deviational variables;

(d) the z_i^* are the desired *goal levels* of achievement for the k criterion functions; and

(e) S is the feasible region $\{x\in\mathbf{R}^n \mid Ax=b, x\geqslant 0\}$.

When there is only one priority level, we can solve the linear goal programming problem using ordinary linear programming. When there is more than one priority level, we have the *preemptive* model which involves the solution of a sequence of LP's, one for each priority level, subject to the optimal deviational variable objective function value of each higher priority LP.

Numerous applications of GP have been reported. See, for example, Lee (with various coauthors) [15,17,18 and 19], Ignizio [9] and Charnes, Cooper and Niehaus [4].

In multiple objective linear programming (Zeleny [23], Evans and Steuer [6] and Isermann [10]) we formulate the MOLP

$$\max \quad \{c^1x=z_1(x)\},$$
$$\max \quad \{c^2x=z_2(x)\},$$
$$\vdots$$
$$\max \quad \{c^kx=z_k(x)\},$$
$$\text{s.t.} \quad x\in S$$

where:

(a) the c^ix are the (linear) criterion functions; and

(b) the $z_k(x)$ are the criterion values as a function of $x\in S$.

In multiple objective linear programming, the strategy is to generate points from the set of efficient solutions in hopes that one of them will be optimal for the decision-maker. Illustrative applications of the MOLP approach appear in Eatman and Sealey [5] and Steuer and Schuler [21].

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With regard to the multiple objective programming in this paper, two slightly different notions of efficiency are defined.

(1) A point $\bar{x} \in S$ is said to be *strongly efficient (s-efficient)* if and only if there does not exist another $x \in S$ such that $z_i(x) \geqslant z_i(\bar{x})$ for all i and $z_i(x) > z_i(\bar{x})$ for at least one i.

(2) A point $\bar{x} \in S$ is said to be *weakly efficient (w-efficient)* if and only if there does not exist another $x \in S$ such that $z_i(x) > z_i(\bar{x})$ for all i.

Note that the set of w-efficient points is a superset of the set of s-efficient points. Although contrived examples can be constructed where the set of w-efficient points is significantly different from the set of s-efficient points [14, Example 2.5], in practice one can expect there to be little if any difference between the two sets.

Recently, multiple objective linear programming has been generalized to include linear fractional (i.e., linear numerator and linear denominator) criteria. In Kornbluth and Steuer [13], an algorithm has been developed for solving the multiple objective linear fractional program (MOLFP)

$$\max \quad \left\{ \frac{c^1x + \alpha_1}{d^1x + \beta_1} = z_1 \right\},$$
$$\vdots$$
$$\max \quad \left\{ \frac{c^kx + \alpha_k}{d^kx + \beta_k} = z_k \right\},$$
$$\text{s.t.} \quad x \in S,$$

where:

(a) the α_i and β_i are constants; and

(b) it is customary to assume that the $d^ix + \beta_i > 0$ for all $x \in S$,

for all w-efficient vertices of the feasible region. The solution set concept of w-efficiency has been adopted because it is more workable [14, Section 2] with multiple fractional criteria than s-efficiency.

The algorithm can be viewed as a generalized MOLP solution procedure in that

(a) the criteria $(c^ix + \alpha_i)/d^ix + \beta_i)$ are linear when $d^i = 0 \in \mathbf{R}^n$; and

(b) the solution set concept of w-efficiency subsumes that of s-efficiency (the normal solution set concept of an MOLP).

Frequently, we will write an MOLFP as

$$\overline{\text{w-eff}} \quad \left\{ \begin{array}{l} \dfrac{c^1x + \alpha_1}{d^1x + \beta_1} = z_1 \\ \vdots \\ \dfrac{c^kx + \alpha_k}{d^kx + \beta_k} = z_k \end{array} \right\}$$
$$\text{s.t.} \quad x \in S,$$

where $\overline{\text{w-eff}}$ (with the overbar) signifies that we are looking for the set of w-efficient solutions in a *maximizing* sense.

The purpose of this paper is to present a generalized approach for solving a goal program with linear fractional criteria. In solving such a GP, we will employ some techniques from (regular) goal programming and some from multiple objective linear fractional programming. The idea is to perform a variable change on the deviational variables d_i^- and d_i^+ and then, except in the case when there is only one criterion associated with each priority level, solve the resulting mathematical program as an MOLFP.

To indicate the usefulness of a linear fractional goal programming capability, Table 1 compares some traditional linear criteria with possible linear fractional criteria that are now facilitated in a GP modeling framework.

2. Fractional objectives and the problems of nonlinearity

Let us assume that some or all of the DM's criterion functions are linear fractional of the form

$$\frac{c^ix + \alpha_i}{d^ix + \beta_i}.$$

As shown in [11], the preemptive GP model can be adapted to the situation in which there is only one criterion function associated with each priority class. In such cases, at any stage j, we are dealing with just one criterion function. If the jth criterion is linear, normal LP methods are used. If the jth criterion is fractional, we can use the single objective linear fractional programming methods of Charnes and Cooper [3], Martos [20] or Bitran and Novaes [1] to solve for the jth goal. This is done as follows.

Table 1
Possible linear fractional criteria

Functional area	Traditional linear criteria	Possible fractional criteria
Financial and corporate planning	profit	debt/equity ratio
	dividends	current ratio
	sales	return on investment
	etc.	etc.
Production planning	costs	inventory/sales
	overtime	actual cost/standard cost
	units produced	output/employee
	etc.	etc.
Marketing and media selection	salesmen allocation	market share
	sales goals	advertising expense/sales
	advertising exposure units	*j*th product sold/sales
	etc.	etc.
University planning and student admissions	students accepted	student/teacher ratio
	class size	cost/student
	new appointments	tenured/non-tenured faculty ratio
	etc.	etc.
Health care and hospital planning	manpower requirements	cost/patient
	patients treated	nurse/patient ratio
	bed-night occupancy	utilization ratios
	etc.	etc.

Without loss of generality, assume $j = 1$. We augment the feasible region S by adding the constraint

$$\frac{c^1x+\alpha_1}{d^1x+\beta_1} + d_1^- - d_1^+ = z_1^*.$$

Since $d^1x+\beta_1 > 0$ (by assumption) we cross-multiply and obtain

$$(c^1 - z_1^* d^1)x + d_1^-(d^1x+\beta_1) - d_1^+(d^1x+\beta_1) = = z_1^*\beta_1 - \alpha_1. \tag{2.1}$$

Note that (2.1) is nonlinear. However, we can use the *variable change*

$$u_i^- = d_i^-(d^1x+\beta_1) \quad \text{and} \quad u_i^+ = d_i^+(d^1x+\beta_1)$$

and formulate our goal programming problem as

$$\begin{aligned} \min \quad & \frac{w_1^- u_1^- + w_1^+ u_1^+}{d^1x+\beta_1}, \\ \text{s.t.} \quad & (c^1 - z_1^* d^1)x + u_1^- - u_1^+ = z_1^*\beta_1 - \alpha_1, \quad (2.2) \\ & x \in S, \\ & x, u_1^-, u_1^+ \geqslant 0, \end{aligned}$$

which is a standard single objective linear fractional programming problem. Note that for each goal i, the w_i^- and w_i^+ are *intragoal* weights. Within goal i, the w_i^- and w_i^+ specify the relative penalties to be applied to under achievements or overachievements from z_i^*. In no way do the w_i^- and w_i^+ attempt to reflect the relative importance among goals.

If the goal z_1^* is attainable then we will have an optimal solution to (2.2) with u_1^- and $u_1^+ = 0$ and S would be augmented with the constraint

$$(c^1 - z_1^* d^1)x = z_1^*\beta_1 - \alpha_1$$

before proceeding to the next priority level. If z_1^* cannot be attained, but rather $\hat{z}_1$ is the nearest possible attainment, then S would be augmented with the constraint

$$(c^1 - \hat{z}_1 d^1)x = \hat{z}_1\beta_1 - \alpha_1.$$

The method, however, is only successful when there is just one fractional criterion per priority class. If there are two such functions, then the problem becomes

$$\begin{aligned} \min \quad & \left\{\lambda_1\left(\frac{w_1^- u_1^- + w_1^+ u_1^+}{d^1x+\beta_1}\right) + \lambda_2\left(\frac{w_2^- u_2^- + w_2^+ u_2^+}{d^2x+\beta_2}\right)\right\}, \\ \text{s.t.} \quad & (c^1 - z_1^* d^1)x + u_1^- - u_1^+ = z_1^*\beta_1 - \alpha_1, \\ & (c^2 - z_2^* d^2)x + u_2^- - u_2^+ = z_2^*\beta_2 - \alpha_2, \\ & x \in S, \\ & x, u^-, u^+ \geqslant 0. \end{aligned} \tag{2.3}$$

In (2.3), the $\lambda_i > 0$ are *intergoal* weights that reflect the relative importance among goals. The λ_i are often difficult for a decision-maker to estimate. Fortunately, in the method of this paper, such intergoal weights do not have to be specified.

The problem with (2.3) is that the sum of two linear fractionals is in general a quadratic (and not a linear) fractional function—and thus single objective linear fractional methods cannot be used. Noting this difficulty with the preemptive model, let us now consider Section 3.

3. Fractional goal programming and multiple objective linear fractional programming

Instead of each fractional criterion being assigned its own priority level, let us assume the non-preemptive case in which all criteria are at the same level. Employing the variable change, we have

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} \lambda_i \left(\frac{w_i^- u_i^- + w_i^+ u_i^+}{d^i x + \beta_i} \right), \\ \text{s.t.} \quad & (c^i - z_i^* d^i)x + u_i^- - u_i^+ = \\ & \quad = z_i^* \beta_i - \alpha_i, \quad i = 1,\dots,k, \\ & x \in S, \\ & x, u^-, u^+ \geqslant 0, \end{aligned} \tag{3.1}$$

which has, at least, linear constraints. The objective function, however, is the sum of linear fractional functions and is, in general, nonlinear. The difficulty of course with (3.1) is that it may have several local optima and may be unsolvable in any kind of reliable fashion.

How then should (3.1) be solved? The method proposed is to use multiple objective linear fractional programming because of the ability of an MOLFP algorithm [13] to characterize the set of all possible optimal solutions (see the Appendix) by computing all w-efficient vertices.

Let

$$v_i(x) = \frac{w_i^- u_i^- + w_i^+ u_i^+}{d^i x + \beta_i}$$

and consider the MOLFP

$$\begin{aligned} \underline{\text{w-eff}} \quad & \{v_1(x), v_2(x), \dots, v_k(x)\}, \\ \text{s.t.} \quad & \text{constraints of (3.1)} \end{aligned} \tag{3.2}$$

where $\underline{\text{w-eff}}$ (with the underbar) signifies that we are looking for w-efficient solutions in a *minimizing* sense. (Note that $\underline{\text{w-eff}}\{v(x)\}$ is the same as w-eff$\{-v(x)\}$). As shown in the Appendix, the global optimum of (3.1) will be in the neighborhood of solutions characterized by all w-efficient vertex solutions of (3.2). To find the optimum (or a sufficiently close approximation to terminate the decision process) we can employ the intra-set point generation and filtering devices described in [22].

The use of an MOLFP algorithm has an additional advantage that we can combine fractional goals and fractional objectives in one model. This is accomplished as follows.

Suppose we have k fractional criterion functions, f of which have goals z_i^* and $(k-f)$ of which are to be maximized. Augmenting the constraint set and performing the variable change for the f fractional goals, we have the MOLFP

$$\begin{aligned} \text{w-eff} \quad & \{-v_1(x), \dots, -v_f(x), z_{f+1}(x), \dots, z_k(x)\}, \\ \text{s.t.} \quad & (c^i - z_i^* d^i)x + u_i^- - u_i^+ = \\ & \quad = z_i^* \beta_i - \alpha_i, \quad i = 1,\dots,f, \\ & x \in S, \\ & x, u^-, u^+ \geqslant 0. \end{aligned} \tag{3.3}$$

The set of solutions to (3.3) will contain *both* the fractional goal programming optimum (for goals $1,\dots,f$) *and* the entire set of w-efficient attainments with respect to both the goal deviations ($v_i(x)$) and the objectives ($z_i(x)$). As shown in [12] the updated tableau at a w-efficient vertex $\bar{x}$ can be used to determine the relative importance of the criteria in terms of the intergoal weights λ_i at that vertex.

Let R be the matrix of nonbasic reduced costs pertaining to the updated tableau at w-efficient vertex $\bar{x}$ in (3.3). Paralleling the reasoning associated with Theorem 1 in Geoffrion [7], $\bar{x}$ will be locally optimal for

$$\begin{aligned} \max \quad & \left\{ \sum_{i=1}^{f} -\lambda_i v_i(x) + \sum_{i=f+1}^{k} \lambda_i z_i(x) \right\}, \\ \text{s.t.} \quad & \text{constraints of (3.3)}, \\ & \sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i \geqslant 0 \end{aligned}$$

for some vector of intergoal weights λ.

Using R, $\bar{x}$ will be locally optimal for any intergoal weighting vector λ provided it satisfies

$$\lambda^T R \geqslant 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i \geqslant 0.$$

Thus we can specify, if so desired, the set of intergoal weighting vectors associated with a particular w-efficient vertex. We note that these weights are not supplied a priori *by* the decision-maker—they are supplied a posteriori by a model *to* the decision-maker. Recall that the only weights that have to be set a priori are the intragoal weights w_i^- and w_i^+ for each goal fractional criterion.

4. Numerical example

Consider the linear fractional GP

$$\left\{\frac{x_1-4}{-x_2+3}=z_1(x)\right\},$$

$$\left\{\frac{-x_1+4}{x_2+1}=z_2(x)\right\},$$

$$\{-x_1+x_2=z_3(x)\},$$

$$\begin{aligned} \text{s.t.}\quad -x_1+3x_2 &\leqslant 0,\\ x_1 \qquad &\leqslant 6,\\ x &\geqslant 0, \end{aligned}$$

where:

(a) $z^*=(z_1^*,z_2^*,z_3^*) \quad =(1,2,0)$,

(b) $w^-=(w_1^-,w_2^-,w_3^-)=(100,200,1)$,

(c) $w^+=(w_1^+,w_2^+,w_3^+)=(200,100,1)$.

This problem might describe a highly simplified production problem with two decision variables x_1 and x_2. Criterion z_3 might be profit and z_1 and z_2 some operating ratios. The goal $z_3^*=0$ implies that we are looking for a breakeven point. The intragoal weighting structure implies that it is twice as important to minimize the overattainment of goal 1 as opposed to minimizing its underattainment. The opposite is true for goal 2.

Augmenting the constraint set along with the variable change, we have the non-preemptive GP

$$\min\left\{\lambda_1\left(\frac{100u_1^-+200u_1^+}{-x_2+3}\right)+\lambda_2\left(\frac{200u_2^-+100u_2^+}{x_2+1}\right)+\lambda_3(d_3^-+d_3^+)\right\},$$

$$\begin{aligned} \text{s.t.}\ -x_1+3x_2 & &&\leqslant 0,\\ x_1 & &&\leqslant 6,\\ x_1+x_2+u_1^--u_1^+ & &&=7,\\ x_1+2x_2 &\quad -u_2^-+u_2^+ &&=2,\\ -x_1+x_2 &\quad +d_3^--d_3^+ &&=0,\\ \text{all variables} &\geqslant 0, && \end{aligned} \tag{4.1}$$

that is graphed in Fig. 1.

The shaded areas and line segments described by $x^1,x^2,\ldots,x^{10}$ constitute the set of all w-efficient solutions. Noting that level curves of the two fractional criteria are straight lines emanating from the 'rotation points' $r^1=(4,3)$ and $r^2=(4,-1)$, respectively, the dashed lines on the figure are the goal lines $z_1^*=1$, $z_2^*=2$ and $z_3^*=0$ along with their associated deviational variables.

Solving the MOLFP

$$\underline{\text{w-eff}}\quad\left\{\begin{array}{c}\dfrac{100u_1^-+200u_2^+}{-x_2+3}\\ \dfrac{200u_2^-+100u_2^+}{x_2+1}\\ d_3^-+d_3^+\end{array}\right\}$$

s.t. constraints of (4.1),

the w-efficient vertices $x^1,x^2,\ldots,x^{10}$ result as listed in Table 2.

Suppose that upon examination of the w-efficient vertices, the decision-maker selects x^2 as his most preferred. At x^2 the reduced cost matrix

$$R=\begin{bmatrix} 88.89 & 100.00 & -33.33 & 33.33 & 0.00\\ 0.00 & 0.00 & 200.00 & 100.00 & 0.00\\ -3.00 & 0.00 & 1.00 & -1.00 & 2.00 \end{bmatrix}$$

where the columns correspond to the nonbasic variables x_2,u_1^+,u_2^-,u_2^+ and d_3^+. Using (3.4), the domain of intergoal weighting vectors λ is given by

$$\begin{aligned} 88.89\,\lambda_1 \qquad\qquad -3\lambda_3 &\geqslant 0,\\ 100.00\,\lambda_1 \qquad\qquad &\geqslant 0,\\ -33.33\,\lambda_1+200\lambda_2+\lambda_3 &\geqslant 0,\\ 33.33\,\lambda_1+100\lambda_2-\lambda_3 &\geqslant 0,\\ 2\lambda_3 &\geqslant 0,\\ \lambda_1+\lambda_2+\lambda_3 &=1,\\ \lambda_1,\lambda_2,\lambda_3 &\geqslant 0. \end{aligned} \tag{4.2}$$

For instance, $\lambda_1=\lambda_2=\lambda_3=1/3$ makes x^2 optimal in (4.2). Other values such as $\lambda_1=0,\lambda_2=1,\lambda_3=0$ in which the second goal is of paramount importance, and $\lambda_1=7/6,\lambda_2=1/7,\lambda_3=0$ in which considerable emphasis is placed upon the first goal also satisfy (4.2). The space of λ's for which x^2 is optimal is shown in Fig. 2 as the hatched area superimposed upon the triangle.

With regard to the intergoal weighting vectors w^- and w^+ (which in the numerical example of this section equal (100,200,1) and (200,100,1) re-

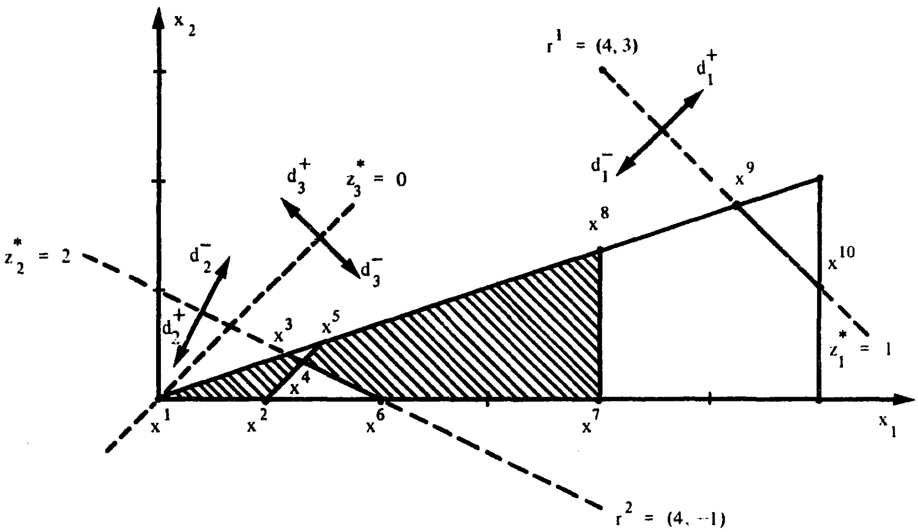

Fig. 1. Graph of numerical example.

Table 2
W-efficient vertices of numerical example

w-efficient vertex	coordinates	z_1	z_2	z_3	Deviational variables		
x^1	(0,0)	$-4/3$	4	0	$d_1^- = 7/3$	$d_2^+ = 2$	
x^2	(1,0)	-1	3	-1	$d_1^- = 2$	$d_2^+ = 1$	$d_3 = 1$
x^3	(6/5,2/5)	$-14/13$	2	$-4/5$	$d_1 = 27/13$		$d_3 = 4/5$
x^4	(4/3,1/3)	-1	2	$-5/4$	$d_1 = 2$		$d_3 = 5/4$
x^5	(3/2,1/2)	-1	5/3	-1	$d_1 = 2$	$d_2 = 1/3$	$d_3 = 1$
x^6	(2,0)	$-2/3$	2	-2	$d_1 = 5/3$		$d_3 = 2$
x^7	(4,0)	0	0	-4	$d_1 = 1$	$d_2 = 2$	$d_3 = 4$
x^8	(4,4/3)	0	0	$-8/3$	$d_1 = 1$	$d_2 = 2$	$d_3 = 8/3$
x^9	(21/4,7/4)	1	$-5/11$	$-7/2$		$d_2 = 27/11$	$d_3 = 7/2$
x^{10}	(6,1)	1	-1	-5		$d_2 = 3$	$d_3 = 5$

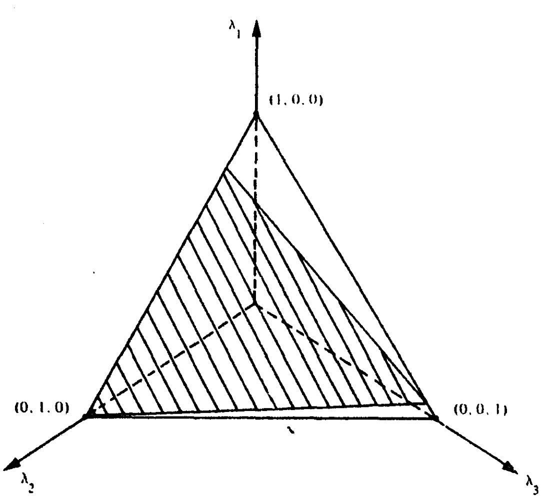

Fig. 2. The domain of λ for which x^2 is w-efficient.

spectively) we note without loss of generality that all of the entries in these vectors can be set to either 0 or 1. In case of a less than or equal to goal i, we could set $w_i^- = 0$ and $w_i^+ = 1$. With a greater than or equal to goal i, we can set $w_i^- = 1$ and $w_i^+ = 0$. And with an equality goal i, we can set $w_i^- = w_i^+ = 1$. Of course the intergoal λ_i weights computed in (4.2) would be different because the contents of the reduced cost matrix R would be different, but the w-efficient solutions generated by the fractional GP procedure would be the same.

5. Concluding remarks

Section 2 discussed a method for solving pre-emptive linear fractional goal programs and Section 3 presented a method for solving non-preemptive linear fractional goal programs. When using the method of Charnes and Cooper [3] at each stage, the advantage of the preemptive approach is that it can be solved using regularly available software. The advantage of the non-preemptive approach is that more than one fractional goal can be accommodated at a priority level and a priori intergoal weights are not required.

Although these are two different approaches, hybrid models are envisioned. Suppose for example that we have a preemptive GP with five linear fractional criteria: one at the 1st priority level, one at the 2nd, two at the 3rd and one at the 4th.

We would begin by following the preemptive approach for the first two priority levels. Then all of the remaining goals (the two from the 3rd priority level and the one from the 4th priority level) would be grouped into a single formulation and the preemptive model would be followed to complete the solution of the problem.

Thus we have a flexible range of approaches consisting of the preemptive, hybrid and non-preemptive methods, none of which require the explicit specification of intergoal weights, for solving goal programs with linear fractional criteria.

Appendix

An MOLFP algorithm is used to solve the fractional goal programming problem because the optimal solution of an fractional GP is w-efficient.

Consider the two problems

$$\text{w-}\overline{\text{eff}} \quad \{f_1(x),\dots,f_k(x) | x \in S\}, \tag{A1}$$

$$\max \quad \sum_{i=1}^{k} \lambda_i f_i(x), \tag{A2}$$

$$\text{s.t.} \quad x \in S,$$

in which

(a) $\lambda_i > 0$ for all i; and

(b) the $f_i(x)$ are real valued functions defined over S.

In connection with these problems, a point $\bar{x} \in S$ is *properly efficient* if and only if

(a) $\bar{x}$ is s-efficient; and

(b) there exists an $M > 0$ such that for all $x \in S$ with $f_i(x) > f_i(\bar{x})$ there is some index $j \neq i$ for which $f_j(x) < f_j(\bar{x})$ with

$$\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leq M.$$

From Theorem 1 in Geoffrion [7] we know that if $\bar{x}$ is optimal in (A2), $\bar{x}$ is properly efficient in (A1). Since any point that is properly efficient is also s-efficient and any point that is s-efficient is w-efficient, all properly efficient points are w-efficient. Thus the reason for using an MOLFP algorithm to solve a non-preemptive GP because the set of all properly efficient points can be characterized by the computation of all w-efficient vertices.

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