On the Dynamics of Occupational Choice, Human Capital and Inequality *

Maximiliano Dvorkin† Alexander Monge-Naranjo†§

March 21, 2019

Preliminary

get the latest version here

Abstract

We develop a tractable dynamic Roy model in which infinitely-lived workers choose occupations to maximize their lifetime utility. In our setting, a worker's human capital is driven by his labor market choices, given idiosyncratic occupation-specific productivity shocks and the costs of switching occupations. We characterize the equilibrium assignment of workers to jobs and show that the resulting evolution of aggregate human capital across occupations ultimately determines the long-run rate of growth of the economy. We then use our model to quantitatively study the dynamic impact of labor-saving technical changes, e.g. automation, on the workers' occupational choices, and on the economy's income inequality, job polarization and long-run growth.

* Very preliminary and incomplete. We thank Ariel Burstein, Paco Buera, Loukas Karabarbounis, Dirk Krueger, Rody Manuelli, Paulina Restrepo, Andres Rodriguez-Clare, Esteban Rossi-Hansberg, Kim Ruhl, Juan Sanchez, Yongs Shin, and seminar and conference participants for helpful comments. All views and opinions expressed here are the authors' and do not necessarily reflect those of the Federal Reserve Bank of St. Louis or the Federal Reserve System. † *Federal Reserve Bank of St. Louis.* § *Washington University in St. Louis.*

1 Introduction

Technological and organizational advances that can greatly expand production possibilities are often biased against a subset of occupations –and, possibly, against a large number of workers. The introduction of such advances can substantially disrupt the ongoing reassignment of workers across jobs, a process that occurs naturally according to the comparative advantage of each worker. Labor market disruptions of this nature have been highlighted by recent works –which we discuss below– that single out the introduction of new forms of capital, e.g., computers and robots, automation, and off-shoring, as the key drivers of the observed increase in earnings inequality and job polarization. Basing their analysis on static Roy models, these papers capture how individual comparative advantage and self-selection shape the heterogeneous impact across workers, but abstract from dynamic aspects of occupational choices and human capital accumulation, which, as we show in this paper, are crucial to determine the the long-run rate of growth rate of the economy and the ultimate impact on inequality and on the welfare of workers.

In this paper, we develop a dynamic Roy model of occupational choice with human capital accumulation and use it to explore the general equilibrium effects of new technologies on the labor market. In our model, infinitely-lived workers can switch occupations in any period to maximize their lifetime utility. In our setting, a worker's human capital is driven by his labor market choices, given idiosyncratic occupation-specific productivity shocks and the costs of switching occupations. We first characterize the equilibrium assignment of workers to jobs. A key result is that the resulting evolution of aggregate human capital across occupations ultimately determines the long-run rate of growth rate of the economy. We then use the model to quantitatively study how worker's individual occupation choices change with the introduction of new technologies, and in turn how this choices shape the equilibrium allocation of workers to different jobs, the dynamics of aggregate human capital, the behavior of earnings inequality, the evolution of the labor share, and the welfare of the different workers in the economy.

The paper has a number of methodological contributions. First, we fully characterize the solution of the recursive problem of a worker under standard CRRA preferences when the worker is subject to a large number of labor market opportunities shocks in every period affecting her comparative advantage in different occupations. Thus, we bridge recent quantitative work that uses static assignment Roy models with extreme-value shocks with the standard recursive models for households in macroeconomics. In this way, our model generates transition probabilities across occupations over time. Second, we fully characterize the asymptotic behavior of aggregate economies implied by the individual dynamic occupation choices of workers. For any given vector of skill prices, we show that the economy converges to a unique invariant distribution of workers. Although the Roy model has been studied and used in great length, we uncover important new features which are present only in a dynamic context. We show that, generically, the reallocation of workers to occupations combined with the accumulation of occupational human capital leads to sustained growth over time for the economy. The growth rate in our model is endogenously determined by the equilibrium occupational choices, and thus, changes in economic conditions that alter worker's choices affect the long-run growth rate of the economy. Third, we embedded the workers' problem in a fairly rich general equilibrium environment where different types of workers are allocated to different tasks in production. We derive a very transparent and tractable aggregation that arises from the assignment of workers to tasks. Then, we show the existence of a competitive-equilibrium balanced-growth path, and for a simple version of our model we can also characterize uniqueness. Fourth, by incorporating two forms of physical capital, we provide a quantitative framework to study the impact of automation and other labor-saving technological improvements on the earnings of different occupations. Our model of production and tasks generates an intuitive expression that directly links the labor share of the economy with wages, rental rates and the productivity of different types of labor and capital, allowing us to study the effects of technology on the labor share of the economy. Fifth, we extend recent dynamic-hat-algebra methods and show they can be used with more general preferences (CRRA) and with human capital accumulation. As with other hat-algebra methods, the advantage is a substantially reduced set of calibrated parameters needed for the quantitative application of the model. Sixth, we discuss a variety of relevant extensions of our baseline model, ranging from workers' age and ex-ante heterogeneity, endogenous on-the-job training and occupation-specific automation.

Using our model we make a number of substantial contributions. Mapping our model to the moments observed in the 1970s for the U.S. economy, we account for the changes in employment across occupations and the increase in earnings inequality that arise from labor-saving technological advances. An important change observed in U.S. labor markets in the past few decades is the polarization of skills in the labor market. That is, the decline of employment in middle-skill occupations, like manufacturing and production occupations, and the growth of employment in both high and low-skill occupations, like managers and professional occupations on one end, and assisting or caring for others on the other. Using our model we show how some labor-saving technical improvements can jointly explain the increase in polarization, earnings inequality and occupational mobility in U.S. labor markets.

In addition, our dynamic model highlights the long-lasting impact of permanent, but once-and-for-all technological changes. Indeed, in our dynamic setting, once-and-for-all changes in automation or other technological changes can lead to sustained growth effects. Our quantitative exercise highlight how this growth effect changes the conclusion on earnings inequality and welfare. We emphasize that the welfare and inequality implications for technological changes can be vastly richer than those obtained in other settings as they originate not only from changes in skills prices in each period but also on changes in the equilibrium growth rate of earnings. Thus, on the one hand, the positive impact on some workers is not only due to higher level of earnings but also from a faster growth. On the other hand, some workers can be worse-off due to lower levels of earnings and a higher rate at which they change occupations. These aspects are fully incorporated in our exercises.

We first consider the individual worker's problem. Section 2 studies the dynamic problem of a worker that chooses occupations to maximize lifetime utility. Taking as given a vector of skill prices or unitary wages per occupation, we assume that each period each worker is subject to idiosyncratic labor market opportunities. On the basis of these labor market opportunities, the choice of occupations not only determines the earnings for the next period but also the impact of the human capital of the worker for subsequent occupation choices. Assuming standard CRRA preferences and extreme value (Frechet) distributed labor market opportunities, we characterize the Bellman equation of the worker. We show that the resulting distribution of the value function is closely related to one of the three extreme value distributions, Frechet, Gumbel or Weibull, depending on whether the coefficient of relative risk aversion is lower than, equal to or higher than one, respectively. In all these cases, conditions for existence and uniqueness are provided. Simple recursion formulas ensue, which makes for trivial computations. The worker's decision problem induces very simple formulas for the transition probabilities of workers across human capital and the law of motion for individual human capital and earnings, which we later use to calibrate the model.

From the worker's individual occupation choices we derive the law of motion for the population of workers across occupations. For any given positive vector of skills prices, we show that there exists a unique invariant distribution of workers. More interestingly, we also show that a simple aggregation property holds, which allows us to write down the transition matrix for the vector of aggregate human capital across occupations. We show that the human capital of the country cannot settle down to an invariant state, and instead, necessarily, grows over time. The dominant eigenvalue of the aggregate human capital transition matrix is always unique, positive and real and it governs the long-run growth rate of the economy. In other words, we show that our dynamic Roy model with human capital accumulation provides a novel endogenous channel for aggregate growth. We examine how changes in the relative price of skills leads to differences in the assignment of workers to jobs, the evolution of human capital and in turn the long-run growth of the economy.

In Section 3 of the paper we embed the previous analysis in a dynamic general equilibrium model. We assume that output is produced using two forms of physical capital. The first physical capital is in the form of structures and other capital complementary to workers. The second form of physical capital is in the form of machines or some types of equipment, which directly compete and may substitute workers in production. In our setting, output is produced by performing a large set of tasks. An assignment of workers and machines to the different tasks is presented, where the costs of the different factors, relative to their underlying productivity, governs which tasks are performed by machines and which are done by different types of workers. We characterize the labor-share of the economy as a simple function of wages in different occupations, rental rates, and labor and machine productivity and show how changes in this variable affect the labor share. The equilibrium production assignment of tasks-workers-machines gives rise to a transparent and very tractable aggregation of the economy. Over time, the accumulation of both forms of capital are determined by standard Euler equations, as in the neoclassical growth model. More novel, however, given a constant productivity terms for both capital stocks and for the vector of human capital, the underlying long-run growth rate of the economy is determined by the transition matrix of human capital, as derived from the dynamic occupation choices of workers. We show the existence, and in a simple case uniqueness, of the competitive-equilibriumn balanced-growth path (BGP) of the economy. As noted above, the growth rate is endogenously determined by the Perron root of the transition matrix, and thus, it endogenously changes with permanent but once-and-for all changes in the vector of relative productivities. Thus, changes in the economy's growth rate lie at the heart of the impact of automation and other technological changes.

In Section 4 we examine the dynamic response of the economy to changes in the productivity of machines and workers in different occupations. Here, we extend the recent dynamic hat algebra methods to a larger set of preferences and to human capital accumulation. The main advantage of these methods is that they avoid the estimation or calibration of a large number of parameters, and instead use moments that can be readily retrieved from the data, lowering the burden of the computational problem.

1.1 Related Literature:

Our work is related to a large literature in labor economics and macroeconomics studying the effects of labor-saving technology in the labor market. This literature has been carefully summarized in the work of Acemoglu and Autor (2011). They analyze how changes in technology may have an asymmetric impact on workers, leading to polarization and earnings inequality. In particular, they argue that "recent technological developments have enabled information and communication technologies to either directly perform or permit the offshoring of a subset of the core job tasks previously performed by middle skill workers, thus causing a substantial change in the returns to certain types of skills and a measurable shift in the assignment of skills to tasks". They propose task-based framework for analyzing the allocation of skills to tasks and for studying the effect of new technologies on the labor market and their impact on the distribution of earnings. In our model workers human capital and skills evolve endogenously and are a result of past labor market decisions. We study the dynamics of adjustment of an economy to an increase in labor-saving technologies affecting workers' occupational decisions, human capital accumulation and earnings inequality.

Krusell, Ohanian, Rios-Rull, and Violante (2000) study how skill biased technical change affect the skill premium and earnings inequality. They argue that the sharp reduction in the price of equipment coupled with differences in capital-skill complementarity are responsible for a large fraction of the increase in inequality between education groups. In their paper, labor markets are segmented by education and workers cannot switch their type. In our case, we follow a task-approach with labor markets segmented at the level of occupations. In our model, workers have different skills and a comparative advantage in performing different tasks, they accumulate human capital, and can switch to a different labor market. In this way, workers have a way to "escape" the negative effects of technology but at a cost in term of a human capital depreciation.

Kambourov and Manovskii (2009) argue that wage inequality and occupational mobility are intimately related. They show this using a general equilibrium model with occupation-specific human capital and two economies with different levels of occupational mobility. Relative to them, we follow a task-based approach, where workers have a comparative advantage in performing different tasks. In our setup, workers search over labor markets is directed and workers self-select into the most valuable occupations. We analyze the effects of a technology shock and compute transition between balanced-growth paths.

Our dynamic Roy model model extends an important recent literature using static Roy model with Frechet distributed shocks building in the setup by Eaton and Kortum (2002). Some recent examples include Lagakos and Waugh (2013), who analyze the assignment of workers to rural or urban work, Hsieh, Hurst, Jones, and Klenow (2013) who study how frictions in the labor market across workers with different characteristics (race, gender) generate misallocation and productivity costs, and Burstein, Morales, and Vogel (2018), who use an assignment model with many labor groups, equipment types, and occupations, in which changes in inequality are driven by the asymmetric impact of changes in the workforce composition, the occupation demand, and new technologies. Methodologically, we extend this type of models to a dynamic context where workers face a trade-off between staying in a low paying occupation or switching to a better labor market at the expense of a mismatch of their human capital.

In addition, we connect to the recent works by Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2017) who study how machines and industrial robots affect different workers and labor markets. In addition, they argue that part of the decline in the labor share discussed in Karabarbounis and Neiman (2013) may be explained by this type of technological advances. Acemoglu and Restrepo (2018) propose a task-based approach between labor and machines to analyze how new technologies may displace labor from some tasks. We extend their setup to different types of labor and show how model of production delivers a a very intuitive expression for the labor share that can be directly linked to the evolution of fundamentals.

Finally, we extend the recent dynamic-hat-algebra methods of Caliendo, Dvorkin, and Parro (2019) to allow for more general preferences and for human capital accumulation. Moreover, we show existence of the competitive general-equilibrium balanced-growth path, and in a simpler case we also show uniqueness of the general equilibrium.

2 A Canonical Worker's Problem

We consider an infinite horizon maximization problem for a worker with standard preferences. At any time $t = 0, 1, 2, ...$, the utility of the worker is given by

$$U_t = \frac{(c_t)^{1-\gamma}}{1-\gamma} + E\left[\sum_{s=1}^{\infty} \beta^s \frac{(c_{t+s})^{1-\gamma}}{1-\gamma}\right],$$

where $0 < \beta < 1$ is a discount factor (which accounts for a constant survival probability) and $\gamma \geq 0$ is the coefficient of relative risk aversion (CRRA.) For $\gamma = 1$, we interpret the flow utility to be logarithmic, i.e. $\ln c_t$.

The worker starts each period $t = 1, 2, ...$ attached to one of $j = 1, ..., J$ occupations, carrying over from the previous period an *absolute level* of human capital $h > 0$. Available for the next period, the worker realizes a vector $\epsilon_t = \left[\epsilon_t^1, \epsilon_t^2, ...\epsilon_t^\ell ... ,\epsilon_t^J\right] \in \mathbb{R}_+^J$ of labor market opportunities. Each entry in the vector corresponds the labor market opportunity in the respective occupation. On the basis of these opportunities, the worker chooses to either stay in the current occupation j or to move to an alternative occupation ℓ.

Switching occupations entails costs (or returns) which we specify as follows: A $J \times J$ *human capital transferability* matrix, with strictly positive entries, $\tau_{j\ell}$, determines the fraction of the human capital h that can be transferred from the current occupation j to a new occupation ℓ. On average, there is depreciation if $\tau_{j\ell} < 1$ or positive accumulation if $\tau_{j\ell} > 1$. The diagonal terms, τ_{jj}, may be higher than one, capturing learning-by-doing, i.e. the accumulated experience capital of a worker as he spends more time in an occupation j. These diagonal terms τ_{jj} may vary by occupation j. The off-diagonal terms $\tau_{j\ell}$ may be less than one to capture a potential mismatch between the human capital acquired in one occupation and the productivity of the worker in a different occupation. Still, some of the off-diagonal terms could be greater than one, capturing skill transferability and cross-occupation training or upgrading. In our specification, these occupation-switching costs have a permanent impact on the human capital of the worker for all future periods and for all future occupation choices.[1]

The human capital of the worker evolves according to the labor market opportunities ϵ_t and the occupation choice of the worker. Given a level of human capital, h, a current occupation j, and a vector $\epsilon_t \in \mathbb{R}_+^J$ of labor market opportunities, the vector

$$h_t \, \tau_{j,\cdot} \otimes \epsilon_t \in \mathbb{R}_+^J,$$

describes how many efficient units of labor services, or effective human capital, the worker can provide for each of the alternative occupations $\ell = 1, ..., J$. Here the operator $\otimes$ denotes an element-by-element multiplication. After choosing which occupation to take, the scale of the

[1] In Appendix B we extend the model to allow for both, transitional and permanent costs of switching occupations.

human capital level for the worker for the next period would be

$$h_{t+1} = h_t \tau_{j_t, \ell_{t+1}} \epsilon_{\ell,t}, \tag{1}$$

where j_t and ℓ_{t+1} indicate, respectively, the occupations at period t and $t+1$.

To set up our framework, in this section we focus on the canonical worker's problem given a time-invariant vector of strictly positive (and finite) wages per unit of human capital $w = \left[w^1, w^2, ... w^\ell ..., w^J\right]$. Therefore, the worker's earnings for the period given her current occupation j are $w^j h_t$. In our model, workers are dynamic optimizers, with their human capital returns as their sole source of income in every period. Therefore, the worker's consumption for each period is simply the current earnings $w^j h_t$.

We now set up the problem of the worker recursively, and provide additional structure to derive a sharp characterization of the optimal choices. Denote by $V(j, h, \epsilon)$ the expected life-time discounted utility of the worker. The Bellman Equation (BE) that defines this value function is,

$$V(j, h, \epsilon) = \frac{\left(w^j h\right)^{1-\gamma}}{1-\gamma} + \beta \max_\ell \left\{E_{\epsilon'} V\left[\ell, h', \epsilon'\right]\right\}, \tag{2}$$

where $E_{\epsilon'}[\cdot]$ is the expectation over the next period's vector of job market opportunities and h' is given by equation (1.)

To characterize this BE, we first show that it can be *factorized*, i.e. its value can be decomposed into a factor that depends only on the current occupation and labor market realizations, (j, ϵ), and another factor that depends only on the absolute level of human capital, h. This can be done for any generic distribution for the labor market shocks ϵ for which an expectation satisfies a boundedness condition. *In all what follows, we assume that ϵ is distributed independently over time and across workers, and that all the required moments involving ϵ are finite and well defined.*

First, note that if occupation ℓ is chosen, then, the next period human capital is $h' = h\, \tau_{j\ell} \epsilon_\ell$. Then, observe that the period utility function is homogeneous of degree $1-\gamma$ in h. Therefore, under the hypothesis that the value $V(j, h, \epsilon)$ is homogeneous of degree $1-\gamma$ in h, for any pair (j, ϵ), it can be factorized into a real value $v(j, \epsilon)$ and a human capital factor $h^{1-\gamma}$, i.e., $V(j, h, \epsilon) = v(j, \epsilon)\, h^{1-\gamma}$.[2] Under this hypothesis, the Bellman Equation (2) becomes

$$v(j, \epsilon)\, h^{1-\gamma} = \left(\frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta \max_\ell \left\{E_{\epsilon'}\left[v\left(\ell, \epsilon'\right)\right] \left(\tau_{j,\ell} \epsilon^\ell\right)^{1-\gamma}\right\}\right) h^{1-\gamma}.$$

Simplifying out the term $h^{1-\gamma}$ we end up with

$$v(j, \epsilon) = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta \max_\ell \left\{\left(\tau_{j,\ell} \epsilon^\ell\right)^{1-\gamma} E_{\epsilon'}\left[v\left(\ell, \epsilon'\right)\right]\right\}, \tag{3}$$

[2] For the logarithmic case, $\gamma = 1$, $V(j, h, \epsilon) = u_j + \beta\left[\max_\ell \left\{v_\ell + \frac{[\ln(h) + \ln(\tau_{j,\ell}\epsilon_\ell)]}{1-\beta}\right\}\right]$, as shown in the appendix.

which verifies the factorization hypothesis. Therefore, the characterization of $V(j,h,\epsilon)$ boils down to the characterization of $v(j,\epsilon)$, a random variable that depends on each realization ϵ. For all occupations $j=1,...J$, denote by v^j the conditional expectation of this random variable, i.e.,

$$v^j \equiv E_\epsilon \left[v(j,\epsilon)\right].$$

Using this definition, and taking the expectation $E_\epsilon[\cdot]$ in both the right- and left-hand sides of (3), the equation reduces to a recursion on v^j:

$$v^j = \begin{cases} \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta E_\epsilon \max_\ell \left[\left\{\left[\tau_{j,\ell}\,\epsilon^\ell\right]^{1-\gamma} v^\ell\right\}\right], & \text{for } \gamma \neq 1, \\ \\ \ln w^j + \beta E_\epsilon \left[\max_\ell \left\{v^\ell + \frac{\ln\left(\tau_{j,\ell}\epsilon^\ell\right)}{1-\beta}\right\}\right], & \text{for } \gamma = 1. \end{cases} \tag{4}$$

For all $\gamma \geq 0$, the following lemma establishes simple conditions on the stochastic behavior of the labor market opportunities of workers, that guarantee the existence and uniqueness of values $v \in \mathbb{R}^J$ that solve (4). All along, we assume that $\tau_{j,\ell} > 0$ for all j,ℓ and that the support of ϵ^ℓ is $[0,\infty)$ for all ℓ.

Depending on the value of γ, and for each $j=1,...,J$, we define the terms, Φ_j as follows:

$$\Phi_j \equiv \begin{cases} E_\epsilon \max_\ell \left\{\left[\tau_{j,\ell}\,\epsilon_\ell\right]^{1-\gamma}\right\}, & \text{for } 0 \leq \gamma < 1, \\ E_\epsilon \max_\ell \left\{\ln\left(\tau_{j,\ell}\epsilon_\ell\right)\right\}, & \text{for } \gamma = 1. \\ E_\epsilon \min_\ell \left\{\left[\tau_{j,\ell}\,\epsilon_\ell\right]^{1-\gamma}\right\}, & \text{for } \gamma > 1. \end{cases}$$

Also, conditioning on the relevant definition of Φ_j for each γ, we define

$$\bar{\Phi} = \max_j \Phi_j.$$

The following lemma shows that if the average labor market opportunities available to workers are bounded, as summarized by bounds on $\bar{\Phi}$, then, we can guarantee that the dynamic programming problem (4) has a unique and well-defined solution.

Lemma 1 *Let $w \in \mathbb{R}^J_+$ be the vector of unitary wages across all occupations J. Assume that preferences are characterized by a CRRA $\gamma \geq 0$ and that the matrix $\tau_{j,\ell}$ and labor market opportunities ϵ satisfy the assumptions above. Then:* ***(a)*** *for all $0 < \gamma \neq 1$, if $\beta\bar{\Phi} < 1$, then there exists a unique, finite $v \in \mathbb{R}^J$ that solves $v^j = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta E_\epsilon \max_\ell \left[\left\{\left[\tau_{j,\ell}\,\epsilon^\ell\right]^{1-\gamma} v^\ell\right\}\right]$ for all j. Moreover, if $\gamma < 1$, the fixed point v is positive ($v \in \mathbb{R}^J_+$) and if $\gamma > 1$, the fixed point v is negative ($v \in \mathbb{R}^J_-$).* ***(b)*** *For the special case of log preferences, $\gamma = 1$, if $-\infty < \Phi_j < +\infty$, $\forall j$, and $\beta < 1$, then, there exists a unique, finite $v \in \mathbb{R}^J$ such that $v^j = \ln w^j + \beta E_\epsilon \left[\max_\ell \left\{v^\ell + \frac{\ln\left(\tau_{j,\ell}\epsilon^\ell\right)}{1-\beta}\right\}\right]$ for all j.*

The proofs for this and all other analytical results in this paper are in Appendix A.

This lemma only verifies existence and uniqueness of the conditional expectations v^j, while the realization ϵ of the labor market opportunities determines the actual realized value $v(j,\epsilon)$. In what follows we impose additional structure so we can characterize the behavior of $v(j,\epsilon)$ and the optimal occupation choices derived for the solution to the dynamic programming problem of workers. To this end, we add the assumption that each element in the vector of labor market opportunities ϵ is distributed according to an extreme value distribution. Specifically, we impose that in each period, *the labor market opportunity ϵ^ℓ shocks for each labor market ℓ, are each independently distributed according to a Frechet distribution with scale parameter $\lambda_\ell > 0$, and curvature $\alpha > 1$.* Notice that the curvature parameter is the same for all occupations but the scale parameters are allowed to vary across occupations.

Having impossed a Frechet distribution for ϵ, define for all pairs $j,\ell \in J \times J$,

$$\Omega_{j\ell} = \begin{cases} \tau_{j\ell}^{(1-\gamma)} v^\ell, & \text{for } \gamma \neq 1, \\ \\ \frac{\ln \tau_{j\ell}}{1-\beta} + v^\ell, & \text{for } \gamma = 1. \end{cases} \tag{5}$$

Then, the normalized BE can be succinctly rewritten as

$$v(j,\epsilon) = \begin{cases} u^j + \beta \max_\ell \left\{ \Omega_{j\ell} \left(\epsilon^\ell\right)^{1-\gamma} \right\}, & \text{for } \gamma \neq 1, \\ \\ u^j + \beta \max_\ell \left\{ \Omega_{j\ell} + \frac{\ln\left(\epsilon^\ell\right)}{1-\beta} \right\}, & \text{for } \gamma = 1. \end{cases} \tag{6}$$

We now provide a simple result that indicates that given any admissible vector $v \in \mathbb{R}^J$, regardless of whether it is or not the fixed point of the BE (4), the resulting random variable $v(j,\epsilon)$ is closely related to one of the extreme value distributions: *(a)* if $0 \leq \gamma < 1$, then $v(j,\epsilon)$ is related to a *Frechet* with curvature parameter $\alpha/(1-\gamma)$; *(b)* if $\gamma = 1$, then $v(j,\epsilon)$ is related to a *Gumbel* with shape parameter $1/\alpha$; *(c)* if $\gamma > 1$, then $v(j,\epsilon)$ is related to a *Weibull* with curvature parameter $\alpha/(\gamma-1)$.

Lemma 2 ***Derived Distributions.*** *Let ϵ^ℓ be a random variable distributed Frechet with scale parameter $\lambda_\ell > 0$ and curvature $\alpha > 1$, i.e. its c.d.f. is $F_\epsilon(\epsilon) = e^{-\left(\frac{\epsilon}{\lambda_\ell}\right)^{-\alpha}}$. Define:*

$$x_\ell \equiv \begin{cases} \left(\epsilon^\ell\right)^{1-\gamma} & \textit{for } 0 \leq \gamma \neq 1 \\ \ln\left(\epsilon^\ell\right) & \textit{for } \gamma = 1. \end{cases}$$

Then x_ℓ is distributed as follows:

$$x_\ell \sim \begin{cases} \textit{Frechet}\left(\frac{\alpha}{1-\gamma}, (\lambda_\ell)^{1-\gamma}\right) & \textit{for } 0 \leq \gamma < 1, \\ \textit{Gumbel}\left(\frac{1}{\alpha}, \ \ln(\lambda_\ell)\right) & \textit{for } \gamma = 1, \\ \textit{Weibull}\left(\frac{\alpha}{\gamma-1}, \ (\lambda_\ell)^{\gamma-1}\right) & \textit{for } \gamma > 1. \end{cases}$$

It follows that the terms in curly brackets in (6), with the product of Ω and the transformation of ϵ with respect to the CRRA parameter, will follow one these distributions.

We now complete the characterization of the worker's problem, under the assumption that all the entries of ϵ are independently Frechet distributed. The following theorem provides a simple, sharp characterization for the fixed point problem v^j that solves (4) and for the optimal occupations decision of workers.

Theorem 1 ***Individual Problems.*** *Assume for all $\ell = 1, ...J$, the shocks ϵ_ℓ are independently distributed Frechet with shape $\alpha > 1$ and scales $\lambda_\ell > 0$. Assume also that all w^ℓ are strictly positive and that either (i) $\gamma \neq 1$ and $\beta\bar{\Phi} < 1$ or (ii) $\gamma = 1$, $\beta < 1$ and $-\infty < \Phi_j < +\infty$ for all j. Then:*
(i) *If $0 \leq \gamma < 1$, the expected values v^j for $j = 1, ..., J$ solve the fixed point problem*

$$v^j = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta\,\Gamma\left(1 - \frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^{J}\left(v^\ell\right)^{\frac{\alpha}{1-\gamma}}\,\left(\tau_{j\ell}\lambda_\ell\right)^\alpha\right]^{\frac{1-\gamma}{\alpha}}.$$

A finite solution $v \in \mathbb{R}_+^J$ for this BE exists and is unique. Moreover, the proportion of workers switching from occupation j to occupation ℓ at the end of the period is given by:

$$\mu\left(j,\ell\right) = \frac{\left[\lambda_\ell\tau_{j\ell}\left(v^\ell\right)^{\frac{1}{1-\gamma}}\right]^\alpha}{\sum_{k=1}^{J}\left[\lambda_k\tau_{jk}\left(v^k\right)^{\frac{1}{1-\gamma}}\right]^\alpha}.$$

(ii) *If $\gamma = 1$ the expected values v^j for $j = 1, ..., J$, solve the fixed point problem*

$$v^j = \log(w^j) + \frac{\beta}{\alpha\left(1-\beta\right)}\log\left[\sum_{\ell=1}^{J}\exp\left(\alpha\left(1-\beta\right)v^\ell + \alpha\log(\tau_{j\ell}) + \alpha\log(\lambda_\ell) + \alpha\,\kappa\right)\right],$$

where κ is Euler's constant. A solution $v \in [\underline{v}, \bar{v}]^J$ for this BE exists and is unique. Moreover, the proportion of workers that switch from occupation j to occupation ℓ is given by:

$$\mu\left(j,\ell\right) = \frac{\exp\left(\alpha\left(1-\beta\right)v^\ell + \alpha\log(\tau_{j\ell}) + \alpha\log(\lambda_\ell) + \alpha\,\kappa\right)}{\sum_{k=1}^{J}\exp\left(\alpha\left(1-\beta\right)v^k + \alpha\log(\tau_{jk}) + \alpha\log(\lambda_k) + \alpha\,\kappa\right)}.$$

(iii) *If $\gamma > 1$, the expected values v^j for $j = 1, ..., J$ solve the fixed point problem*

$$v^j = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} - \beta\,\Gamma\left(1 - \frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^{J}(-v^\ell)^{\frac{\alpha}{1-\gamma}}\,\left(\tau_{j\ell}\lambda_\ell\right)^\alpha\right]^{\frac{1-\gamma}{\alpha}}.$$

A solution $v \in \left[\underline{v}, 0\right]^J$ for this BE exists and is unique. Moreover, the proportion of workers *switching from occupation j to occupation ℓ at the end of the period is given by:*

$$\mu(j,\ell) = \frac{\left[\lambda_\ell \tau_{j\ell} \left(-v^\ell\right)^{\frac{1}{1-\gamma}}\right]^\alpha}{\sum_{k=1}^J \left[\lambda_k \tau_{jk} \left(-v^k\right)^{\frac{1}{1-\gamma}}\right]^\alpha}.$$

2.1 Implied Distributions of Workers and Human Capital

We now describe how the occupation choices of each worker shape up the limiting behavior of the cross-section distribution of workers and aggregate human capital (and earnings) across the J occupations in this economy.

Distribution of Workers Across Occupations. Notice that the homogeneity of the value function implies that the transitions $\mu_{j,\ell}$ are independent of the human capital level h of the worker. Let $\theta_t = \left[\theta_t^1, ..., \theta_t^J\right]$ denote the $J \times 1$ vector indicating the mass of workers in each of the occupations $j = 1, 2, ..., J$ at time t. As in this section we are taking the vector of wages w as time invariant the transition matrix μ is also time invariant. Therefore, the evolution of θ is described by following equation,

$$\theta_{t+1} = \mu^T \theta_t.$$

where superscript T indicates the transpose.

Under the assumptions that $\tau_{j,\ell} > 0$, every entry of the stochastic matrix μ is positive, i.e. for all j, ℓ, $\mu(j,\ell) > 0$. This is a basic mixing condition and from standard results for Markov chains (e.g. Theorem 11.2. in Stokey, Lucas and Prescott, 1989) there exists a unique invariant distribution

$$\theta_\infty = \mu^T \theta_\infty, \tag{7}$$

and the economy will converge to it from any initial distribution θ_0.

Distribution of Aggregate Human Capital Across Occupations. Given that the individual labor market opportunities or productivity shocks for all workers are distributed Frechet, a continuous distribution with full support in the positive reals, then the aggregate human capital assigned to occupation j is given by,

$$H_t^j = \theta_t^j \int_0^\infty h \phi_t^j(dh),$$

where $\phi_t^j(\cdot)$ denotes the positive measure that describes the distribution of human capital levels h across the workers in occupation j in period t.

Characterizing the evolution of H_t^j over time suffices to determine the general equilibrium of the economy as we discuss in the following section. Towards that end, we first characterize the conditional expectation of the shocks ϵ^ℓ for those workers that switch from any occupation j to any occupation ℓ:

Lemma 3 *For all non-negative $\gamma \neq 1$, the expectation of the labor market opportunity shock ϵ_ℓ of* *workers switching from j to ℓ is given by,*

$$E\left[\epsilon_\ell | \Omega_{j\ell}\,\epsilon_\ell^{1-\gamma} = \max_k \left\{\Omega_{j\ell}\epsilon_k^{1-\gamma}\right\}\right] = \Gamma\left(1-\frac{1}{\alpha}\right)\lambda_\ell\left[\mu\left(j,\ell\right)\right]^{-\frac{1}{\alpha}}, \tag{8}$$

where $\mu(j,\ell)$ is the corresponding occupation switching probabilities as derived in Theorem 1.

A worker with human capital h in occupation j will switch to occupation ℓ at the end of the period with probability $\mu(j,\ell)$, bringing an average $\Gamma\left(1-\frac{1}{\alpha}\right)\tau_{j\ell}\,\lambda_\ell\left[\mu(j,\ell)\right]^{-\frac{1}{\alpha}} h$ of human capital skills to that occupation. Define $\mathcal{M}$ to be the transition matrix of aggregate human capital, with j,ℓ element defined as:

$$\mathcal{M}(j,\ell) = \Gamma\left(1-\frac{1}{\alpha}\right)\tau_{j\ell}\,\lambda_\ell\left[\mu(j,\ell)\right]^{1-\frac{1}{\alpha}}.$$

The matrix $\mathcal{M}$ is time invariant when wages are constant over time. The linearity in h allows an easy aggregation of human capital in each occupation and also to characterize the law of motion for aggregate human capital. Let $H_t = \left[H_t^1, H_t^2, ..., H_t^J\right]$ be the vector of aggregate human capital across all occupations j in period t. Then, for time $t+1$, that vector evolves according to

$$H_{t+1} = \mathcal{M}^T H_t.$$

It is worth remarking that we can characterize the evolution of the average (or total) skills of workers across the different occupations, without having to solve for the cross-section distribution of skills and earnings. This result is useful since we can easily solve for the aggregate supply of efficient units of labor in each occupation at each t. The matrix $\mathcal{M}$ is strictly positive, i.e. $\mathcal{M}(j,\ell) > 0$. Then, from the Perron-Frobenius theorem,[3] the largest eigenvalue of $\mathcal{M}$ is always simple (multiplicity one), real and positive. Moreover, the associated eigenvector to this so-called Perron root, which we denote by G_H, has all its coordinates, h^j, $j = 1, ..., J$, strictly positive. Moreover, in the limit, the behavior of all H_t^j will converge to

$$H_{t+1}^j = G_H H_t^j,$$

for all $j = 1, ...J$. This is precisely the definition of a balanced-growth path (BGP) for the vector of aggregate human capital $\{H_t\}_{t=0}^{\infty}$. Notice that the model can naturally generate a positive Perron eigenvalue $G_H > 0$, i.e. sustained growth of the human capital of the workers, even if the unitary wages w^j and the cross-section distribution of workers θ_∞ remains constant, and even if the average realization ϵ^ℓ in each occupation is equal to one. The engine of growth here is that workers continuously select the most favorable labor market opportunities.[4]

[3] See for example, Gantmacher (2000), Theorems 1 and 2 of Ch.XIII, Vol. II, page 53.

[4] This result is reminiscent of the mechanism in the models by Luttmer (2007) and Lucas and Moll (2014) in which selection on favorable realization of idiosyncratic shocks endogenously generates growth at the aggregate level. Note however that in our model it is possible for a worker to get a realization of shocks ϵ below one for all components, which implies that human capital will decrease for this individual if $\tau \leq 1$.

We summarize the results for the implied population dynamics of workers and human capital aggregates, $\{\theta_t, H_t\}_{t=0}^{\infty}$ in the following proposition.

Proposition 2 *Assume that the unitary wage vector is strictly positive, $w \in \mathbb{R}^J_{++}$, and that the conditions for Theorem 1 hold. Then:* ***(a)*** *There exists a unique invariant distribution of workers, i.e., $\theta_\infty = \mu\,\theta_\infty$, with $\theta^j_\infty > 0$ and $\sum_{j=1}^J \theta^j_\infty = 1$. Moreover, the sequence $\{\theta_t\}_{t=0}^{\infty}$ induced by (7) converges to θ_∞ from any initial distribution θ_0.* ***(b)*** *There is a unique BGP of aggregate human capital across occupations, $H^j_t/H^1_t = h^j$ for all j, where h^j is equal to the ratios of the j^{th} coordinate to the first coordinate of the Perron eigenvector. Moreover, the economy converges to $H^j_{t+1} = G_H H^j_t$ from any initial vector $H_0 \in \mathbb{R}^J_+$.*

The problem of the worker presented so far can be easily extended to capture worker heterogeneity along permanent characteristics (gender, race, formal education) as well as age. As shown in Appendix B, the setting can be quite flexible in allowing differences in group specific parameters ($\lambda^{\text{group}}_\ell$, $\tau^{\text{group}}_{j\ell}$), thus allowing differentiating between the human capital accumulation that arises from labor market experience from other factors that affect the human capital of workers. Extending the model for age differences would capture differences in the horizon of workers and their dynamic valuation of switching occupations.

In the next section, we embed the workers' problem into a production economy, and show how to extend the results derived here to characterize the general equilibrium of such an environment.

3 The General Equilibrium Model

We now set up our general equilibrium environment. First, we specify the production of final goods, which defines the demand for the different types of labor and capital and the production price of final goods. Second, we define competitive equilibria, where the price of goods, labor services and capital clear all markets. Third, we provide a sharp characterization of the intratemporal equilibrium conditions. Finally, we prove the existence of balance growth path (BGP) equilibria, and discuss the sources of growth in this economy, which includes the sustained accumulation of skills of workers as they switch occupations over their life-cycle.

3.1 The Environment

3.1.1 Production

We consider multiple types of workers and physical capital as factors of production of final goods. Our setting encompasses features of the standard neoclassical model and of recent models of substitution between workers and machines (e.g. Acemoglu and Restrepo (2018)), both of them within a worker-task assignment model (e.g. Costinot and Vogel (2010).) First, as in standard macro models, we allow for some forms of physical capital to operate as a complementary factor of all forms of labor. Second, as in Acemoglu and Restrepo (2018), we also allow some forms of physical capital (machines) to compete with workers in the performance of tasks. As Costinot and Vogel (2010), different types of workers must be be assigned across multiple production tasks according to their comparative advantage, which is determined in general equilibrium. The resulting multi-dimensional production setting allows for technological changes that have a heterogeneous impact on the different types of labor.

Consider an economy with a single final good, which is produced according to a Cobb-Douglas over some forms of physical capital, K_t, which encompasses structures and some forms of equipement, and a bundle of tasks, Q_t,

$$Y_t = (K_t)^{\varphi} (Q_t)^{1-\varphi},$$

where $0 < \varphi < 1$. The bundle of tasks Q_t is given by a CES production function defined over many tasks. The provision of quantities $q_t(x) \geq 0$ for each task x in the continuum $[0, 1]$, give raise to a bundle of tasks Q_t in the amount

$$Q_t = \left(\int_0^1 [q_t(x)]^{\frac{\eta-1}{\eta}} \, dx \right)^{\frac{\eta}{\eta-1}},$$

at time t. The quantity $q_t(x)$ of each task x is performed using different types of labor and/or capital (machines.) In particular, extending the framework of Acemoglu and Restrepo (2018), we assume that machines and all the $j = 1, ...J$ types of labor are *perfect substitutes* to each other in the production of each task x. The production function of $q_t(x)$ is described by

$$q_t(x) = z_t^M(x) M_t(x) + \sum_{j=1}^{J} z_t^j(x) H_t^j(x), \tag{9}$$

where $z_t^j(x)$ is the productivity of labor type j in task x, and $z_t^M(x)$ is the productivity of machines in task x. Here, $H_t^j(x)$ and $M_t(x)$ are total effective units of labor j and machines used in task x.

For tractability, we assume that for all $x \in$ [0,1] and periods t, the productivities of all labor types j and machines, respectively, $z_t^j(x)$ and $z_t^M(x)$ are distributed i.i.d. Frechet. We assume that across all labor types j and machines, the productivity distributions have a common shape parameter $\nu > 1$ and heterogeneous scale parameters $A_t^j > 0$ and $A_t^M > 0$. In this way we can use the results in Eaton and Kortum (2002) to further characterize optimal demands of factors of production for the different tasks and the over production cost of the good as we discuss below.

3.1.2 Capital Owners

We assume a that both forms of physical capital, machines M_t and structures and other equipment K_t are owned by a separate set of households. These households, which we call these households 'capital owners,' have a constant population with measure 1. Capital owners have standard preferences, given by

$$U_t^K = \sum_{s=0}^{\infty} \beta^t \frac{\left(c_t^K\right)^{1-\gamma}}{1-\gamma}, \tag{10}$$

where, for simplicity, we have assumed that the discount factor β and the CRRA γ have the same values as those of the workers, however, we need to assume $\gamma > 0$ for an interior solution on the investment problem.

Capital owners rent out both forms of physical capital to firms, taking as given the rental price of machines, r_t^M and of structures and other equipment, r_t^K. We follow Lucas and Prescott (1971) and Eaton et al (2016) and assume that both forms of capital accumulate over time according to

$$K_{t+1} = \left(1-\delta^K\right) K_t + \xi^K \left(I_t^K\right)^{\varpi_K} \left(K_t\right)^{1-\varpi_K}, \text{ and} \tag{11}$$

$$M_{t+1} = \left(1-\delta^M\right) M_t + \xi^M \left(I_t^M\right)^{\varpi_M} \left(M_t\right)^{1-\varpi_M}. \tag{12}$$

Here, δ^K and δ^M are both in $[0,1]$, and are the depreciation rates of the two forms of capital. The parameters ϖ_K, ϖ_M are both in $(0,1]$, and give raise to curvature in investment, reducing the return to investments I_t^K, I_t^M as they grow relative to the respective the pre-existing capital stocks. Both investments I_t^K, I_t^M are in units of the final good. The strictly positive terms ξ^K and ξ^M capture investment specific productivities.

Capital owners can freely borrow or lend at the gross (real) interest rate R_t. We denote by B_t the net financial position of the representative capital owner in period t. In terms of financial markets, below we consider two polar cases. First, we consider a small open economy in the interest rate R_t in every period is taken exogenous from international capital markets. Second, we consider a closed economy equilibrium in which $B_t = 0$ for all periods.

3.2 Competitive Equilibria

We assume all labor, capital and goods markets are perfectly competitive. Taking as given the sequence $\left\{P_t, w_t^j, r_t^K, r_t^M, R_t\right\}_{t=0}^{\infty}$ of goods prices, the unitary skill price for jobs of all types $j = 1, ..., J$ and the rental rate of both forms of capital, firms and households maximize their current profits and expected lifetime utilities, respectively. To formally define competitive equilibrium in this environment, we first define the individual problems of firms and workers and outline the market clearing conditions.

3.2.1 Workers' Optimization and Labor Supply

The maximization problem of each of the workers is simply the time-varying extension of the problem characterized in Section 2. For brevity, we consider here only the case of $\gamma > 1$, as the other cases are similar. In any event, for every t, j and h, the expected normalized values $\left\{v_t^\ell\right\}_{\ell=1}^J$

solve the problem recursion:

$$v_t^j = \frac{(w_t^j)^{1-\gamma}}{1-\gamma} - \beta\,\Gamma\left(1-\frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^{J}(-v_{t+1}^{\ell})^{\frac{\alpha}{1-\gamma}}\,(\tau_{j\ell}\lambda_\ell)^{\alpha}\right]^{\frac{1-\gamma}{\alpha}}, \tag{13}$$

and the optimal occupation choices, i.e. transitions from any j to any ℓ are given by

$$\mu_t\,(j,\ell) = \frac{\left[\lambda_\ell\tau_{j\ell}\left(-v_{t+1}^{\ell}\right)^{\frac{1}{1-\gamma}}\right]^{\alpha}}{\sum_{k=1}^{J}\left[\lambda_k\tau_{jk}\left(-v_{t+1}^{k}\right)^{\frac{1}{1-\gamma}}\right]^{\alpha}}, \tag{14}$$

where $\left\{v_{t+1}^{\ell}\right\}_{\ell=1}^{J}$ solves the problem for the subsequent period. Similarly, the transition matrix for aggregate human capital from occupation j to occupation ℓ for the time-varying case is simply

$$\mathcal{M}_t\,(j,\ell) = \Gamma\left(1-\frac{1}{\alpha}\right)\tau_{j\ell}\,\lambda_\ell\,[\mu_t\,(j,\ell)]^{1-\frac{1}{\alpha}}\,. \tag{15}$$

The implied laws of motion for the population of workers and aggregate human capital across occupations are, respectively

$$\theta_{t+1} = \mu_t^T\theta_t, \tag{16}$$

and

$$H_{t+1} = \mathcal{M}_t^T H_t, \tag{17}$$

for initially given θ_0 and H_0.

3.2.2 Firms' Optimization and Labor Demand

In this setting, productivity differences and the linearity of $q_t(x)$ ensures that, except for a measure zero, each of the tasks will be provided by only one type of labor j or by only machines, according to their comparative advantage. To see this, let w_t^j be the unitary price of effective labor j and r_t^M be the rental rate of a machine at time t. Because of perfect substitution, the minimum cost of producing $q_t(x)$ units of task x is

$$C_t\,[q\,(x)] = q(x)\times\min\left\{\frac{w_t^1}{z_t^1(x)},\frac{w_t^2}{z_t^2(x)},\ldots,\frac{w_t^J}{z_t^J(x_i)},\frac{r_t^M}{z_t^M(x)}\right\}. \tag{18}$$

Clearly, the ratios between factor prices and productivities determine whether one of the labor types or machines will take care of a particular task.[5] Optimizing firms will minimize the cost of

[5] Acemoglu and Restrepo (2018) considers two factor economies, i.e. machines and one type of labor. Here, we consider a multidimensional setting where cutoffs and the assignments of workers and machines to tasks are randomly determined for tractability.

producing the aggregate bundle of tasks. The unitary cost, C_t^Q, is the solution of the program:

$$C_t^Q = \min_{q_t(x)} \int_0^1 C\left[q_t\left(x\right)\right] dx \quad \text{s.t.} \quad \left(\int_0^1 \left[q_t(x)\right]^{\frac{\eta-1}{\eta}} dx\right)^{\frac{\eta}{\eta-1}} = 1. \tag{19}$$

Finally, given the rental price r_t^K for physical capital K_t, and the unitary cost of tasks C_t^Q, the competitive price of final goods is simply given by

$$P_t = \left[\varphi^{-\varphi}\left(1-\varphi\right)^{\varphi-1}\right]\left(r_t^K\right)^{\varphi}\left(C_t^Q\right)^{1-\varphi}. \tag{20}$$

The next proposition characterizes the solution of the firms' optimization problem:

Proposition 2 *Assume $z_t^j(x)$ and $z_t^M(x)$ are distributed i.i.d. Frechet, all with shape parameter $\nu > 1$ and heterogenous scale parameters: $A_t^j > 0$ for labor type j and $A_t^M > 0$ for machines. Then, for all tasks x, the probability that labor from occupation j implement the tasks is*

$$\pi_t^j = \frac{(w_t^j)^{-\nu}(A_t^j)^{\nu}}{(r_t^M)^{-\nu}(A_t^M)^{\nu} + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^{\nu}}, \tag{21}$$

while the probability that the task is implemented by machines is

$$\pi_t^M = \frac{(r_t)^{-\nu}(A_t^M)^{\nu}}{(r_t^M)^{-\nu}(A_t^M)^{\nu} + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^{\nu}}. \tag{22}$$

The resulting unitary cost of producing the aggregate bundle of tasks, Q_t, is

$$C_t^Q = \Gamma\left(1 + \frac{1-\eta}{\nu}\right)^{\frac{1}{1-\eta}}\left[(r_t^M)^{-\nu}(A_t^M)^{\nu} + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^{\nu}\right]^{-1/\nu}. \tag{23}$$

Moreover, the competitive price the final goods is given by

$$P_t = \Phi_0\left(r_t^K\right)^{\varphi}\left[(r_t^M)^{-\nu}(A_t^M)^{\nu} + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^{\nu}\right]^{\frac{\varphi-1}{\nu}}, \tag{24}$$

where $\Phi_0 \equiv \frac{\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{1-\eta}}}{\varphi^{\varphi}(1-\varphi)^{1-\varphi}} > 0$.

3.2.3 Capital Owners

Given an initial level of both forms of physical capital, $K_0 > 0$, $M_0 > 0$, the initial financial position B_0 and the sequence of good prices, capital rental rates, and interest rates, $\left\{P_t, r_t^K, r_t^M, R_t\right\}_{t=0}^{\infty}$, define the budget constraint, for any period t, as

$$\frac{r_t^M}{P_t}M_t + \frac{r_t^K}{P_t}K_t + R_t B_t = c_t^K + I_t^K + I_t^M + B_{t+1}, \tag{25}$$

where the laws of motion for M_t and K_t are given (12), (11), respectively.

Proposition 3 *Under the conditions just stated, the program of consumption, investments and capital stocks,* $\left\{c_t^K, I_t^K, I_t^M, K_{t+1}, M_{t+1}, B_{t+1}\right\}_{t=0}^{\infty}$, *that maximizes (10) is characterized by a standard transversality condition, and three Euler equations that can be written as:*

$$R_{t+1} = \beta^{-1}\left(\frac{c_{t+1}^K}{c_t^K}\right)^{\gamma}, \tag{26}$$

$$\frac{r_{t+1}^K}{P_{t+1}} = \frac{R_{t+1}\left(\frac{I_t^K}{K_t}\right)^{1-\varpi_K} - \left[(1-\varpi_M)\left(\frac{K_{t+2}}{K_{t+1}}\right) + \left(1-\delta^K\right)\varpi_K\right]\left(\frac{I_{t+1}^K}{K_{t+1}}\right)^{1-\varpi_K}}{\varpi_K \xi^K}, \tag{27}$$

$$\frac{r_{t+1}^M}{P_{t+1}} = \frac{R_{t+1}\left(\frac{I_t^M}{M_t}\right)^{1-\varpi_M} - \left[(1-\varpi_M)\left(\frac{M_{t+2}}{M_{t+1}}\right) + \left(1-\delta^M\right)\varpi_M\right]\left(\frac{I_{t+1}^M}{M_{t+1}}\right)^{1-\varpi_M}}{\varpi_M \xi^M}. \tag{28}$$

Having characterized the individual optimality conditions of all agents in the economy, we now define and characterize the competitive equilibria in this economy.

3.2.4 Competitive Equilibrium

The aggregate demand for each type of labor j, for structures and other equipment, and for machines is as follows: The total payments to workers in occupation j is given by

$$w_t^j H_t^j = (1-\varphi)\,\pi_t^j\,P_t\,Y_t. \tag{29}$$

Similarly, the total payments for the rental of machines is

$$r_t^M M_t = (1-\varphi)\,\pi_t^M\,P_t\,Y_t. \tag{30}$$

Finally, the total payments for the rental of structures and other equipment is

$$r_t^K K_t = \varphi\,P_t\,Y_t.$$

Having laid out the individual optimization problems and the market clearing conditions, we define a competitive equilibrium as follows:

Definition 1 *Given an initial population of workers and their human capital,* $\left\{\theta_0^j,\ H_0^j\right\}_{j=1}^{J}$, *initial stocks of machines and other physical capital* $\{M_0, K_0\}$, *and exogenous sequences* $\left\{A_t^j, A_t^m\right\}_{t=0}^{\infty}$ *an equilibrium is* ***(i)*** *a* price system $\left\{w_t^j, P_t, r_t^K, r_t^M, R_t\right\}_{t=0}^{\infty}$, ***(ii)*** *individual worker occupation* *decisions* $\left\{v_t^j, \mu_t\right\}_{t=0}^{\infty}$, ***(iii)*** *individual firm tasks-allocation choices* $\left\{\pi_t^j, \pi_t^M\right\}_{t=0}^{\infty}$, ***(iv)*** *aggregate* vectors of workers and human capital across occupations, stocks of machines and other physical capital, $\{\theta_t, H_t, M_t, K_t\}_{t=0}^{\infty}$, *and,* ***(v)*** *aggregate output,* worker and human capital reallocations, and flows of investments and of consumption of the owners of capital, $\left\{Y_t,\ \mu_t,\ \mathcal{M}_t, I_t^K, I_t^M, c_t^K\right\}_{t=0}^{\infty}$ *such that:* ***(a)*** *Given* $\left\{w_t^j, P_t, r_t^K, r_t^M\right\}_{t=0}^{\infty}$, *the workers lifetime optimization* $\left\{v_t^j, \mu_t\right\}_{t=0}^{\infty}$ *are given by (13) and (14); the firms optimize production, i.e.* $\left\{\pi_t^j, \pi_t^M, P_t\right\}_{t=0}^{\infty}$ *are given by (21), (22), and (24); and capital owners invest optimally, i.e. according to (25), (27), and (28.)* ***(b)*** *factor markets clear, i.e. (29), (30) hold for every t, and* ***(c)*** *the population of workers and human capital allocation evolve according to (16) and (17 .)*

We now characterize the prices that clear the market in every period given an exogenous level for productivities $\left\{A_t^j, A_t^M\right\}$, and some pre-determined levels of aggregate supplies H_t^j, M_t and K_t.

3.3 Static Market Clearing Conditions

We now consider the intratemporal equilibrium conditions, which, taking as given the period's stock of physical and human capital:

The following proposition characterizes the intratemporal equilibrium conditions:

Proposition 3 ***Aggregation, Intratemporal Equilibrium.*** *Given pre-determined aggregate variables,* $\left\{K_t,\ M_t,\ H_t^j\right\}$, *the intratemporal competitive equilibrium condition imply that the aggregate output of tasks and final goods* $\{Q_t,\ Y_t\}$ *and the equilibrium prices of* $\left\{K_t,\ M_t,\ H_t^j\right\}$, *are given as follows: (a) the total output of bundles of tasks,* Q_t, *is*

$$Q_t = \Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1}{\eta-1}}\left[\left(A_t^M M_t\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell}\left(A_t^\ell H_t^\ell\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1+\nu}{\nu}}; \tag{31}$$

(b) the total output of goods,Y_t, *is*

$$Y_t = \Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{(1-\varphi)}{\eta-1}}\left(K_t\right)^{\varphi}\left[\left(A_t^M M_t\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell H_t^\ell\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{(1+\nu)(1-\varphi)}{\nu}}. \tag{32}$$

(c) The equilibrium real rental rate of capital $\rho_t^K \equiv r_t^K/P_t$, *is*

$$\rho_t^K = \varphi\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left[\left(A_t^M\frac{M_t}{K_t}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell\frac{H_t^\ell}{K_t}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{(1+\nu)(1-\varphi)}{\nu}}. \tag{33}$$

(d) The equilibrium real rental rate of machines $\rho_t^M \equiv r_t^M/P_t$, *is*

$$\rho_t^M \equiv (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K_t}{M_t}\right)^{\varphi}\left[\left(A_t^M\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell\frac{H_t^\ell}{M_t}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A_t^M\right)^{\frac{\nu}{1+\nu}}. \tag{34}$$

(e) The real unitary wages for occupations $j = 1, ..., J$, $\omega_t^j \equiv w_t^j / P_t$,are

$$\omega_t^j = (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K_t}{H_t^j}\right)^{\varphi}\left[\left(A_t^M\frac{M_t}{H_t^j}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell\frac{H_t^\ell}{H_t^j}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A_t^j\right)^{\frac{\nu}{1+\nu}}. \tag{35}$$

These simple aggregation results, which are derived in the appendix, will be used later in our characterization for the equilibrium dynamics of the model. We first examine stationary environments, balanced growth paths, and then examine the dynamic evolution of the economy after a shock that changes the relative productivity of machines and workers of different occupations.

3.3.1 Discussion

Some results are worth highlighting. In particular, the expressions in Proposition 2 together with equilibrium conditions (29) and (30) characterize the labor share of the economy as a function of the levels of technology A, wages and rental rate, a result we highlighted in the introduction. In particular, the labor share of the economy is equal to $1 - \left[(1-\varphi)\pi_t^M + \varphi\right]$. While φ, the share of income devoted to structures in our model, is constant, the share of equipment $(1-\varphi)\pi_t^M$ depends endogenously on technology, wages and the rental rate, and, for example, an increase in A_t^M will lead to a decrease in the labor share. Similar to Acemoglu and Restrepo (2018), the labor share of our economy depends on how efficient are machines in performing differnet tasks relative to labor. In our case, we have several different types of labor, yet the analysis remains tractable.

3.4 Dynamics

We now consider the dynamic behavior of the economy. We first consider the time-invariant equilibria, when the economy follows a balanced-growth paths (BGP). We then consider the behavior of the economy outside the BGP, that is, the dynamic equilibrium responses of the economy to changes in, for example, the underlying productivities of both labor and machines.

3.4.1 Balanced Growth Paths (BGP)

Consider now an economy in which the productivity of factors remain constant over time $A^j > 0$, $A^M > 0$. A time invariant equilibrium would accrue when all the prices of physical and human capital remain constant. The intratemporal equilibrium conditions for real factor prices

$\left(\rho^K, \rho^M, \omega^j\right)$ of all factors must remain constant and satisfy

$$\rho^K = \varphi\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left[\left(A^M\frac{M}{K}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A^\ell\frac{H}{K}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{(1+\nu)(1-\varphi)}{\nu}}, \quad (36)$$

$$\rho^M = (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K}{M}\right)^{\varphi}\left[\left(A^M\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A^\ell\frac{H^\ell}{M}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A^M\right)^{\frac{\nu}{1+\nu}}, \quad (37)$$

$$\omega^j = (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K}{H^j}\right)^{\varphi}\left[\left(A^M\frac{M}{H^j}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A^\ell\frac{H^\ell}{H^j}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A^j\right)^{\frac{\nu}{1+\nu}} \quad (38)$$

where K/M, M/H^ℓ and H^ℓ/H^j are the factor ratios that must remain constant over time.

As shown in the previous section, even with stationary prices, the aggregate human capital in each of the occupations j may grow over time. Hence, instead of looking for steady states, we now characterize the set of balanced-growth paths (BGP) where at constant rates, i.e. for all $j = 1, ..., J$ and $t \geq 0$, we can write:

$$\frac{H^j_{t+1}}{H^j_t} = G_H,$$

for some gross growth $G_H > 0$. The equilibrium growth rate G_H is determined as follows. First, given real wages $\{\omega^j\}_{j=1}^J$, workers solve a time invariant BE, which for $\gamma > 1$ has the form

$$v^j = \frac{(\omega^j)^{1-\gamma}}{1-\gamma} - \beta\,\Gamma\left(1-\frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^{J}(-v^\ell)^{\frac{\alpha}{1-\gamma}}\,\left(\tau_{j\ell}\lambda_\ell\right)^{\alpha}\right]^{\frac{1-\gamma}{\alpha}},$$

exactly as as in Theorem 1. The formulae for μ and $\mathcal{M}$ are the same as in Section 2, and therefore, the growth rate of all forms of human capital H^j will be govern by the Perron root of $\mathcal{M}$, which, as shown there, is unique, real and strictly positive. Second, given a growth rate G_H, the Euler equations (27) and (28) of capital owners require that the rental rates of both forms of physical capital satisfy

$$\rho^K = \frac{\left[R-\left(1-\delta^K\right)\varpi_K-(1-\varpi_K)\left(G_H\right)\right]}{\varpi_K\xi^K}\left[\frac{G_H-\left(1-\delta^K\right)}{\xi^K}\right]^{\frac{1-\varpi_K}{\varpi_K}}, \quad (39)$$

$$\rho^M = \frac{\left[R-\left(1-\delta^M\right)\varpi_M-(1-\varpi_M)\left(G_H\right)\right]}{\varpi_M\xi^M}\left[\frac{G_H-\left(1-\delta^M\right)}{\xi^M}\right]^{\frac{1-\varpi_M}{\varpi_M}}. \quad (40)$$

where we have used that the investment-to-capital ratios consistent with G_H are given by $I^K/K = \left\{\left[G_H-\left(1-\delta^K\right)\right]/\xi^K\right\}^{\frac{1}{\varpi_K}}$ and $I^M/M = \left\{\left[G_H-\left(1-\delta^M\right)\right]/\xi^M\right\}^{\frac{1}{\varpi_M}}$.

Here, we consider two possibilities: *(a) Small Open Economies (SOE):* where the interest rate is exogenously given $R = R^*$. In this case, the consumption of the capital owners will also grow at a constant rate, $c_{t+1}^K/c_t^K = (\beta R^*)^{\frac{1}{\gamma}}$, but this growth rate needs not be equal to the growth rate G_H. *(b) Closed Economies:* In this case, the growth rate G_H also determines the interest rate, $R = \beta^{-1}(G_H)^\gamma$ in equations (39) and (40.)

For concreteness, we use the following definition.

Definition 2 ***A BGP*** *is a vector of factor prices* $\left(\rho^K, \rho^M, \omega^j\right)$*, an interest rate* R*, a growth rate* G_H*, a positive vector* $H \in \mathbb{R}_{++}^J$ *of aggregate human capitals, a positive pair* (K, M) *of physical capital and individual solutions for the workers problems* $\{v, \mu, \mathcal{M}\}$ *such that: (a)* $\left(\rho^K, \rho^M, \omega^j\right)$ *solve the intratemporal conditions (36), (37) and (38) for* H, K, M*; (b) The growth rate* G_H *is the Perron root of* $\mathcal{M}$ *and* H *is the eigenvector associated to that root. (c) Given* G_H *and* H*,* ρ^K, ρ^M *satisfy the Euler equations (39) and (40.) (d) Given* ω^j*,* $\{v, \mu\}$ *solves the individual worker's optimal occupation choice problem and* $\mathcal{M}$ *is the associated transition function for the aggregate human capital. If (1)* $R = R^*$ *for some exogenous* $R^* > 1$*, then the BGP is also* ***BGP equilibrium for a small open economy****. If instead (2)* $R = \beta^{-1}(G_H)^\gamma$*, then the above BGP is also a* ***BGP equilibrium for a closed economy****.*

The simplest case to show existence is for a SOE under the standard investment model, i.e. $\varpi_K = \varpi_M = 1$. In that case the rental rates of both capitals are uniquely pinned down by $\rho^K = \left[R^* - 1 + \delta^K\right]/\xi^K$ and $\rho^M = \left[R^* - 1 + \delta^M\right]/\xi^M$ and independent of the growth rate G_H. In the appendix we show the following:

Theorem 2 *Consider an economy that satisfies the parameter restrictions laid out above. Moreover, assume a constant, strictly positive vector of productivities* $\left(\{A^j\}_{j=1}^J, A^M\right)$*, and that the conditions for Theorem 1 hold. Then:* ***(a)*** *There exist a unique time invariant* $\{v, \mu\}$ *that solve the individual worker's problem.* ***(b)*** *The transition matrix* μ *has a unique invariant distribution of workers, i.e.,* $\theta_\infty = \mu^T \theta_\infty$*, with* $\theta_\infty^j > 0$ *and* $\sum_{j=1}^J \theta_\infty^j = 1$*.* ***(c)*** *There is exists a unique equilibrium BGP.*

The proof for this theorem is in the appendix. For existence and uniqueness it does not matter whether the economy is closed or a small open economy, albeit that distinction can imply important differences in the equilibrium allocations.

3.5 Transitions: Dynamic Hat Algebra

Having established the conditions for a BGP, in this section we examine the implied dynamics of the model outside a BGP. To this end, in this section we extend the Dynamic Hat Algebra (DHA) methods of Caliendo, Dvorkin, and Parro (2019) to a model with general CRRA preferences, human capital accumulation and endogenous growth.

Proposition 3 ***Dynamic Hat Algebra****. Given observed initial allocations of workers and human capital across occupations, observed initial matrix of occupational transition of workers and* *human capital, and observed factor payments, and values for the discount factor, the CRRA coefficient, the curvature parameters* α *and* ν*, we can solve for the sequential equilibrium of this economy in changes towards the Balanced Growth Path. Moreover, given an unanticipated change in machines or workers' productivity, we can compute the sequential equilibrium of this economy in changes towards the new Balanced Growth Path. In both these cases it is not necessary to know the level of other constant parameters.*

The proof of the proposition is in Appendix B.

Using dynamic-hat-algebra methods is particularly convenient for the computation of the model for two reasons. First, the levels of a large set of parameter values, like τ, λ, A are not needed to calibrate the model o to perform counterfactual analysis, only the *changes* in these parameters are required. This implies that the calibration exercise is less demanding. Second, the level of many of the model's endogenous variables are not needed and the initial and terminal values for many endogenous variables expressed in *changes* are easy to characterize.

4 A simple example

We now use a simple example to highlight the main forces at play in our model. We calibrate our economy to three occupations. These occupations differ in the level of wages, the costs of switching occupations in terms of human capital depreciation, and the initial allocation of workers and total human capital.

5 The impact of technology on U.S. labor markets

In this section we conduct the main quantitative exercise. Our goal is to understand to what extent some labor-saving technological advances with an asymmetric impact across occupations can jointly explain the observed trends in U.S. labor markets in terms of labor polarization, earnings inequality, and the labor share. We first start describing the data we use, our initial conditions, the calibration strategy for some parameters and the moments we match.

5.1 Data and initial equilibrium

We assume our economy is initially in a balanced-growth path and we match the initial equilibrium conditions of the model to those observed in the 1970s in the U.S. economy. For this we use different pieces of data. Given our focus on occupations, occupational mobility and earnings dynamics, we need a reliable source of microdata with this information and with a panel dimension. Kambourov and Manovskii (2013) warns about the potential problems in using different available microdata to study occupational mobility. We follow them and use the Panel Study of Income Dynamics

(PSID) which from 1968 to 1980 has corrected some of the problems that typically arise in the report of occupational status and occupational mobility.

The choice of how many occupations to include largely depends, on one hand, on the sample size, and on the other, on computational constraint. We find the first to be more binding given the relatively small sample of the PSID. We calibrate our model to an yearly frequency and to nine occupations: (1) Management, business, and financial operations occupations; (2) Professional and related occupations; (3) Service occupations; (4) Sales and related occupations; (5) Office and administrative support occupations; (6) Construction and extraction occupations; (7) Installation, maintenance, and repair occupations; (8) Production occupations; and (9) Transportation and material moving occupations.[6]

While it is possible to compute the share of workers across occupations θ and the yearly occupational mobility matrix μ directly using shares from the data, i.e. using a bin estimator, we prefer to use an statistical method so our moments are less influenced by the small size of our sample, particularly for some transitions. For this we estimate the transition matrix using the Poisson Maximum-Likelihood methods proposed by Silva and Tenreyro (2006). As we assume that the initial point of our economy is a balanced growth path, the vector θ can be obtained directly from the estimated transition matrix.

The initial value of μ and θ are:

$$\mu_{-1} = \begin{bmatrix} 0.911 & 0.014 & 0.012 & 0.008 & 0.008 & 0.005 & 0.020 & 0.012 & 0.010 \\ 0.025 & 0.879 & 0.015 & 0.010 & 0.011 & 0.006 & 0.026 & 0.016 & 0.012 \\ 0.030 & 0.022 & 0.852 & 0.012 & 0.013 & 0.007 & 0.031 & 0.019 & 0.015 \\ 0.040 & 0.030 & 0.025 & 0.791 & 0.018 & 0.009 & 0.043 & 0.025 & 0.020 \\ 0.039 & 0.029 & 0.024 & 0.016 & 0.800 & 0.009 & 0.041 & 0.024 & 0.019 \\ 0.061 & 0.046 & 0.037 & 0.025 & 0.027 & 0.671 & 0.065 & 0.038 & 0.030 \\ 0.018 & 0.014 & 0.011 & 0.008 & 0.008 & 0.004 & 0.916 & 0.012 & 0.009 \\ 0.029 & 0.022 & 0.018 & 0.012 & 0.013 & 0.007 & 0.030 & 0.856 & 0.014 \\ 0.035 & 0.026 & 0.021 & 0.015 & 0.015 & 0.008 & 0.037 & 0.022 & 0.821 \end{bmatrix}$$

$$\theta_0 = \begin{bmatrix} 0.2338 & 0.1355 & 0.0930 & 0.0466 & 0.0512 & 0.0172 & 0.2599 & 0.0987 & 0.0642 \end{bmatrix}^T.$$

We obtain the matrix $\mathcal{M}$ in a similar way but using information on earnings dynamics for occupational switchers and stayers. Consistent with our model, we assume that the source of earnings growth in the initial balanced growth path is the change in human capital and not the change in unit wages as these are invariant in our setup.[7] Moreover, as is well known, an identification issue arises in the use of Roy models and a normalization is necessary. The problem

[6]We exclude from the analysis farming, fishing, and forestry occupations.

[7]We can easily accommodate a different assumption, allowing for additional sources of growth in our model.

is that with information only on earnings for an initial equilibrium, it is not possible to distinguish the level of unit wages w and the total number of efficiency units of labor (or units of human capital) h across occupations. Thus, we assume that the initial vector of unit wages is equal to one and obtain the matrix $\mathcal{M}$ by the product of the matrix of average earnings changes for occupational switchers and stayers by occupation and the matrix μ, as implied by our model.[8]

$$\mathcal{M}_{-1} = \begin{bmatrix} 0.929 & 0.014 & 0.012 & 0.008 & 0.009 & 0.004 & 0.021 & 0.012 & 0.009 \\ 0.028 & 0.908 & 0.016 & 0.011 & 0.012 & 0.006 & 0.029 & 0.017 & 0.013 \\ 0.032 & 0.023 & 0.871 & 0.012 & 0.014 & 0.007 & 0.034 & 0.019 & 0.015 \\ 0.047 & 0.033 & 0.028 & 0.823 & 0.020 & 0.010 & 0.049 & 0.028 & 0.022 \\ 0.041 & 0.029 & 0.024 & 0.016 & 0.823 & 0.009 & 0.043 & 0.025 & 0.019 \\ 0.068 & 0.048 & 0.040 & 0.026 & 0.030 & 0.694 & 0.072 & 0.041 & 0.032 \\ 0.019 & 0.014 & 0.011 & 0.007 & 0.008 & 0.004 & 0.927 & 0.011 & 0.009 \\ 0.031 & 0.022 & 0.019 & 0.012 & 0.014 & 0.007 & 0.033 & 0.862 & 0.014 \\ 0.039 & 0.028 & 0.023 & 0.015 & 0.017 & 0.009 & 0.041 & 0.023 & 0.831 \end{bmatrix}$$

As expected from the theory, the elements in $\mathcal{M}_{-1}$ resemble hose in μ. Note however, that they differ in important ways. For example, $\mathcal{M}_{-1}$ is not a stochastic matrix. In fact, the rows do not add up to the same number. Also, note that the ratio between $\mathcal{M}$ and μ indicates the expected (average) evolution of human capital for the occupational switchers conditional on switching. This ratio has elements between 0.96 and 1.16.

As we argued the largest eigenvalue of $\mathcal{M}_{-1}$ represents the long-run gross growth rate of this economy, which in this case is 1.023. In other words, our initial balanced growth path has a growth rate of 2.3% per year. Moreover, the normalized vector of aggregate human capital by occupation, H, is,

$$H_0 = \begin{bmatrix} 0.240 & 0.144 & 0.095 & 0.048 & 0.055 & 0.018 & 0.249 & 0.091 & 0.060 \end{bmatrix}^T .$$

As we said before, we assume that the economy is initially in a balanced growth path. We can test this assumption informally by looking at how different the actual data for θ_0 and H_0 is from the ones implied by the respective transition matrices. Figure 1 compares values for θ_0 and H_0 in the data and the ones implied by the balanced-growth path assumption. We can see that the allocations in the data and the ones implied by the mobility matrices are very highly correlated and are of roughly the same magnitude, laying very close to the 45 degree line.

We calibrate the risk aversion parameter γ to 2, an usual value in the macroeconomics literature. Parameter α directly affects the dynamics of earnings and, other things equal, has a direct incidence on the amount of earnings inequality. We assume a value of 25 which implies that permanent earnings shocks at the individual level do not have a large variance, consistent with the empirical literature on earnings dynamics, but over time they accumulate and and are able to

[8]Clearly, this is just a normalization, and thus, inconsequential for the results. Alternatively we could assume that w differs by occupation, and the matrix $\mathcal{M}$ and the vector of human capital would change accordingly.

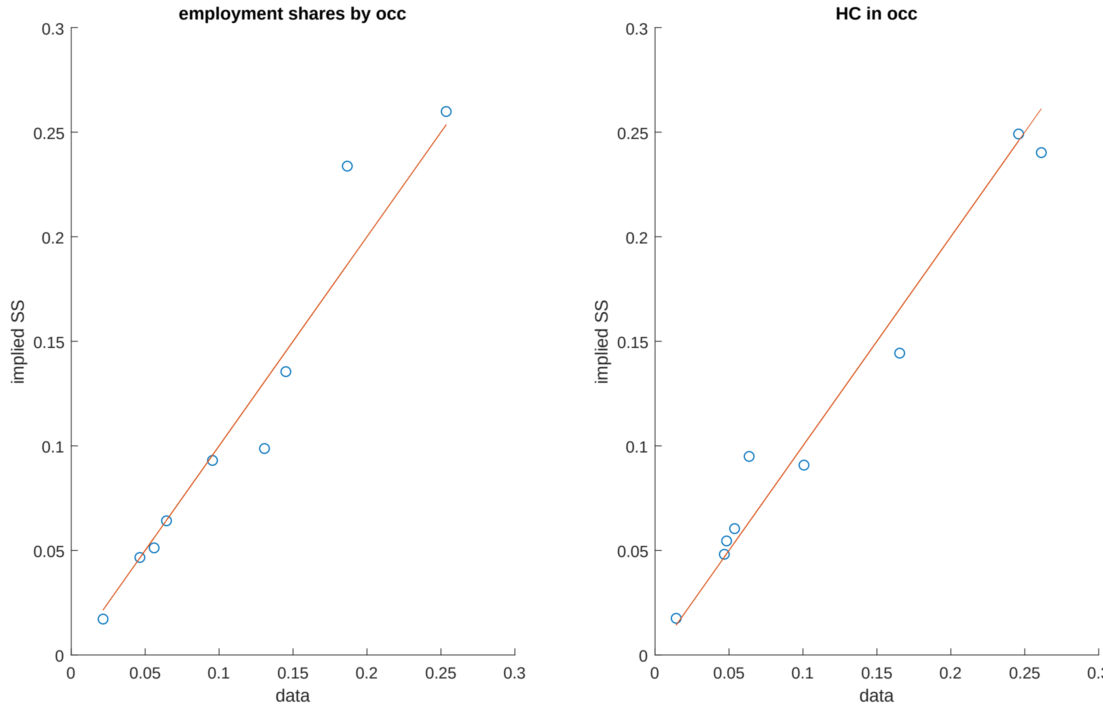

Figure 1: Data vs BGP implied allocations

generate an important amount of inequality in the cross-section. We take calibrate the discount rate β to 0.95.

We use information on National Income and Product Accounts and input output tables to calibrate the share of income going to structures and going to equipment φ and $(1-\varphi)\pi_0^M$. Given our previous assumptions, there is a direct link between the initial shares of income by occupations $(1-\varphi)\pi_0^j$ and the aggregate human capital by occupation H_0^j. We do not have a clear strategy to identify the value for ν, which governs the degree dispersion in productivity of different types of labor and machines in the production of tasks. We calibrate ν to 4 and later analyze how our results change as we vary this value.

5.2 The effects of an asymmetric labor-saving technology shock

We calibrate the technology as a change in A_t^M. We first observe that the equilibrium conditions in our model, in particular the Euler equation for the investment in equipment, (28), imply an inverse relationship between the rental rate r_t^M and investment-specific productivity ξ_t^M. Thus, we follow the literature on skill-biased technical change and use data on the price of investment in equipment relative to the price of consumption to calibrate the evolution of labor-saving technology. However, in stead of using and exogenous change in ξ_t^M, we note that the ξ_t^M and r_t^M are connected in the model, and the way r_t^M and A_t^M are connected in (22), allows us to map the technological change to A_t^M.

We note, however, that for technology to have an asymmetric effect in the labor market, the ratio of A_t^j and A_t^M must vary in a different way across the different occupations. We use information of the share of earnings by occupations to discipline this ratio using the calibrated series of A_t^M directly from the data (changes in the inverse of the relative price of investment).

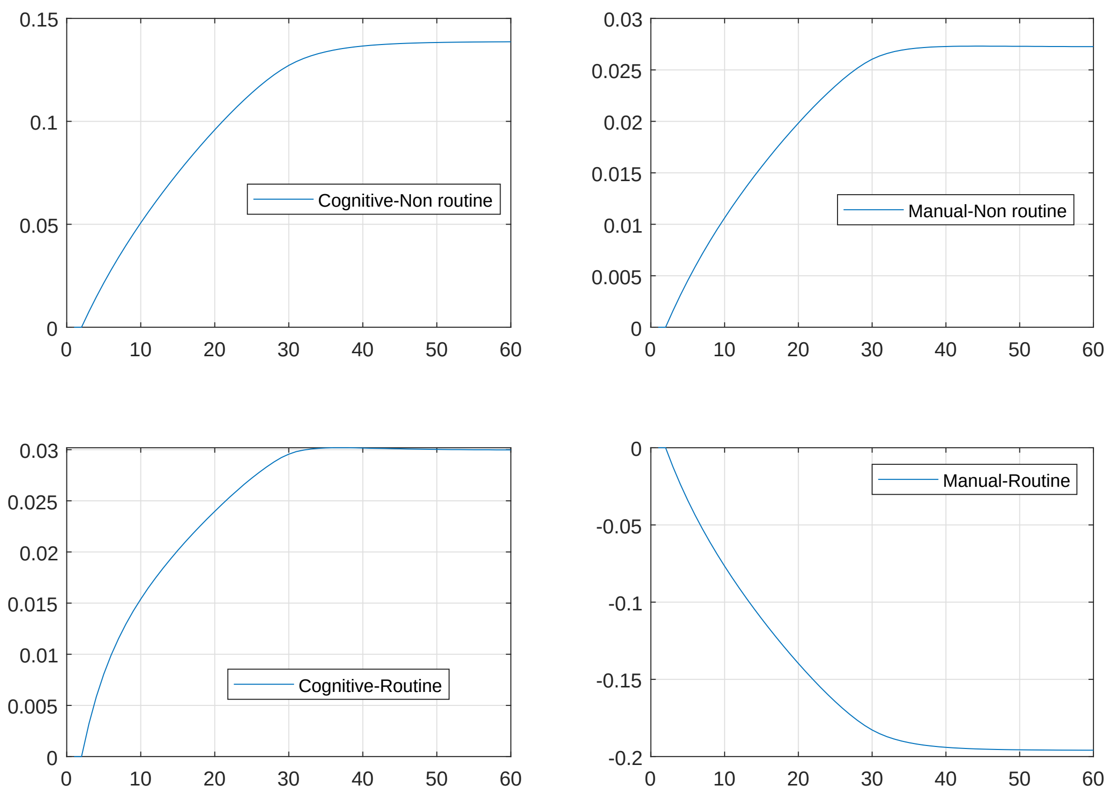

Figure 2: Evolution of employment shares by broad occupation groups

Changes in equilibrium wages and lifetime utility due to the shock, induce a reallocation of workers across occupations. Figure shows the changes in the employment shares by broad occupation groups given the calibrated labor-saving technology shock. The largest impact is in routine-manual occupations, composed of construction, installation and repair, transportation and production, with a sharp reduction in employment. While employment increases in all other occupations, the increase in by far more pronounced in non-routine-cognitive occupations, which include management, professional and technical occupations.

Moreover, and as we emphasized in the Introduction and Section 2, the growth rate of the economy depends on the reallocation. Once the economy stabilized in its new balanced growth path, the economy grows at a gross rate of 1.024. That is, one decimal point higher growth than initially.

6 Conclusion

We develop a dynamic Roy model of occupational choice with human capital accumulation and use it to explore the general equilibrium effects of new technologies on the labor market. In our model, infinitely-lived workers can switch occupations in any period to maximize their lifetime utility. In our setting, a worker's human capital is driven by his labor market choices, given idiosyncratic occupation-specific productivity shocks and the costs of switching occupations. We first characterize the equilibrium assignment of workers to jobs. A key result is that the resulting evolution of aggregate human capital across occupations ultimately determines the long-run rate of growth rate of the economy. We then use the model to quantitatively study how worker's individual occupation choices change with the introduction of new technologies, and in turn how this choices shape the equilibrium allocation of workers to different jobs, the dynamics of aggregate human capital, the behavior of earnings inequality, the evolution of the labor share, and the welfare of the different workers in the economy.

The paper has a number of methodological contributions. First, we fully characterize the solution of the recursive problem of a worker under standard CRRA preferences when the worker is subject to a large number of labor market opportunities shocks in every period affecting her comparative advantage in different occupations. Thus, we bridge recent quantitative work that uses static assignment Roy models with extreme-value shocks with the standard recursive models for households in macroeconomics. In this way, our model generates transition probabilities across occupations over time. Second, we fully characterize the asymptotic behavior of aggregate economies implied by the individual dynamic occupation choices of workers. For any given vector of skill prices, we show that the economy converges to a unique invariant distribution of workers. Although the Roy model has been studied and used in great length, we uncover important new features which are present only in a dynamic context. We show that, generically, the reallocation of workers to occupations combined with the accumulation of occupational human capital leads to sustained growth over time for the economy. The growth rate in our model is endogenously determined by the equilibrium occupational choices, and thus, changes in economic conditions that alter worker's choices affect the long-run growth rate of the economy. Third, we embedded the workers' problem in a fairly rich general equilibrium environment where different types of workers are allocated to different tasks in production. We derive a very transparent and tractable aggregation that arises from the assignment of workers to tasks. Then, we show the existence of a competitive-equilibrium balanced-growth path, and for a simple version of our model we can also characterize uniqueness. Fourth, by incorporating two forms of physical capital, we provide a quantitative framework to study the impact of automation and other labor-saving technological improvements on the earnings of different occupations. Our model of production and tasks generates an intuitive expression that directly links the labor share of the economy with wages, rental rates and the productivity of different types of labor and capital, allowing us to study the effects of technology on the labor share of the economy. Fifth, we extend recent dynamic-hat-algebra methods and show they can be used with more general preferences (CRRA) and with human capital accumulation. As with other hat-algebra methods, the advantage is a substantially reduced set of calibrated parameters needed for the quantitative application of the model. Sixth, we discuss a variety of relevant extensions of our baseline model, ranging from workers' age and ex-ante heterogeneity, endogenous on-the-job training and occupation-specific automation.

Using our model we make a number of substantial contributions. Mapping our model to the moments observed in the 1970s for the U.S. economy, we account for the changes in employment across occupations and the increase in earnings inequality that arise from labor-saving technological advances. An important change observed in U.S. labor markets in the past few decades is the polarization of skills in the labor market. That is, the decline of employment in middle-skill occupations, like manufacturing and production occupations, and the growth of employment in both high and low-skill occupations, like managers and professional occupations on one end, and assisting or caring for others on the other. Using our model we show how some labor-saving technical improvements can jointly explain the increase in polarization, earnings inequality and occupational mobility in U.S. labor markets.

In addition, our dynamic model highlights the long-lasting impact of permanent, but once-and-for-all technological changes. Indeed, in our dynamic setting, once-and-for-all changes in automation or other technological changes can lead to sustained growth effects. Our quantitative exercise highlight how this growth effect changes the conclusion on earnings inequality and welfare. We emphasize that the welfare and inequality implications for technological changes can be vastly richer than those obtained in other settings as they originate not only from changes in skills prices in each period but also on changes in the equilibrium growth rate of earnings. Thus, on the one hand, the positive impact on some workers is not only due to higher level of earnings but also from a faster growth. On the other hand, some workers can be worse-off due to lower levels of earnings and a higher rate at which they change occupations. These aspects are fully incorporated in our exercises.

References

Acemoglu, D., & Autor, D. (2011). Skills, tasks and technologies: Implications for employment and earnings. In *Handbook of labor economics* (Vol. 4, pp. 1043–1171). Elsevier.

Acemoglu, D., & Restrepo, P. (2017). Robots and jobs: Evidence from U.S. labor markets. *Working Paper*.

Acemoglu, D., & Restrepo, P. (2018). The race between man and machine: Implications of technology for growth, factor shares, and employment. *American Economic Review*, *108*(6), 1488–1542.

Burstein, A., Morales, E., & Vogel, J. (2018). Changes in between-group inequality: computers, occupations, and international trade. *American Economic Journal: Macroeconomics*.

Caliendo, L., Dvorkin, M., & Parro, F. (2019). Trade and labor market dynamics: General equilibrium analysis of the China trade shock. *Econometrica, forthcoming*.

Costinot, A., & Vogel, J. (2010). Matching and inequality in the world economy. *Journal of Political Economy*, *118*(4), 747–786.

Eaton, J., & Kortum, S. (2002). Technology, geography, and trade. *Econometrica*, *70*(5), 1741–1779.

Hsieh, C.-T., Hurst, E., Jones, C. I., & Klenow, P. J. (2013). The allocation of talent and U.S. economic growth. *NBER working paper*.

Kambourov, G., & Manovskii, I. (2009). Occupational mobility and wage inequality. *The Review of Economic Studies*, *76*(2), 731–759.

Kambourov, G., & Manovskii, I. (2013). A cautionary note on using (march) current population survey and panel study of income dynamics data to study worker mobility. *Macroeconomic Dynamics*, *17*(1), 172–194.

Karabarbounis, L., & Neiman, B. (2013). The global decline of the labor share. *The Quarterly Journal of Economics*, *129*(1), 61–103.

Krusell, P., Ohanian, L., Rios-Rull, J.-V., & Violante, G. (2000). Capital-skill complementarity and inequality: A macroeconomic analysis. *Econometrica*, *68*(5), 1029–1053.

Lagakos, D., & Waugh, M. E. (2013). Selection, agriculture, and cross-country productivity differences. *American Economic Review*, *103*(2), 948–80.

Lucas, R. E., & Moll, B. (2014). Knowledge growth and the allocation of time. *Journal of Political Economy*, *122*(1), 1–51.

Luttmer, E. G. (2007). Selection, growth, and the size distribution of firms. *The Quarterly Journal of Economics*, *122*(3), 1103–1144.

Silva, J. S., & Tenreyro, S. (2006). The log of gravity. *The Review of Economics and Statistics*, *88*(4), 641–658.