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An analytical derivation of the efficient surface in portfolio selection with three criteria

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Abstract In standard mean-variance bi-criterion portfolio selection, the efficient set is a frontier. While it is not yet standard for there to be additional criteria in portfolio selection, there has been a growing amount of discussion in the literature on the topic. However, should there be even one additional criterion, the efficient frontier becomes an efficient surface. Striving to parallel Merton's seminal analytical derivation of the efficient frontier, in this paper we provide an analytical derivation of the efficient surface when an additional linear criterion (on top of expected return and variance) is included in the model addressed by Merton. Among the results of the paper there is, as a higher dimensional counterpart to the 2-mutual-fund theorem of traditional portfolio selection, a 3-mutual-fund theorem in tri-criterion portfolio selection. 3D graphs are employed to stress the paraboloidic/hyperboloidic structures present in tri-criterion portfolio selection.

Keywords Multiple criteria optimization · Tri-criterion portfolio selection · Minimum-variance frontier · *e*-Constraint method · Efficient surface · Paraboloids

1 Introduction

In investments, portfolio selection is the problem of allocating a given sum of money to securities drawn from a designated pool of securities for the purpose of maximizing the

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future return of the portfolio thus formed, or, that is, for the purpose of maximizing portfolio return. With portfolio return a random variable, the foundation for the solution of the problem of portfolio selection was laid out by Markowitz (1952) in the form of his famous mean-variance model. In this model, "mean" refers to the endeavor to maximize the expected return of the portfolio return random variable, and "variance," which is Markowitz's measure for risk, refers to the endeavor to minimize the variance of the portfolio return random variable. Hence, the so-called mean-variance model is a bi-criterion optimization problem with, as its two objectives, variance to be minimized and expected return to be maximized.

While mean-variance has maintained its status as the predominant model in portfolio selection for over sixty years, it has not been without attempts to extend its scope. One such attempt arose in the 1970s. It is the attempt to include in the portfolio selection process a criterion beyond expected return and variance. Lee (1972) proposed taking dividends into account along with expected return and variance when constructing a portfolio. Stone (1973) proposed skewness as a different kind of third criterion possibility. With only occasional articles on this following, one such being by Spronk et al. (1981), the idea of additional criteria in portfolio selection essentially remained on the back burner of portfolio selection until the mid-1990s when, for instance, Konno and Suzuki (1995) revisited skewness, Chow (1995) considered tracking error as a third criterion, and Speranza (1996) and others mentioned in different ways transaction costs. Soon several survey-type articles appeared such as by Spronk and Hallerbach (1997), Bana e Costa and Soares (2001), and Steuer and Na (2003) giving further impetus to the idea of additional objectives.

While skewness and tracking error are difficult to implement because of their nonlinearities, additional criteria that can be modeled linearly are much more tractable. Recognizing this, the literature then saw a string of contributions involving additional linear criteria and activity on this in the literature has only been steadily increasing. For example, Lo et al. (2003) examined liquidity in this regard, Hallerbach et al. (2004) considered social responsibility, and Ehrgott et al. (2004) took into account the star ranking of a mutual fund. In the list contained in Steuer et al. (2007), the amount invested in R&D (see Guerard and Mark 2004) and growth-in-sales (Ziemba 2006) are also enumerated as possible additional criteria. On the methodological front of how to handle additional criteria in portfolio selection, there are among others the offerings by Ben Abdelaziz et al. (2007), Xidonas et al. (2012), Hirschberger et al. (2013), and Aouni et al. (2014).

However, as of most recently, the additional criterion that appears to be attracting the most attention is social responsibility. Another term often used interchangeably with social responsibility is sustainability. Over the past few years many papers have been written about social responsibility in portfolio selection including those by Ballestero et al. (2012), Dorfleitner et al. (2012), Bilbao-Terol et al. (2013), Calvo et al. (2014), Cabello et al. (2014), Pérez-Gladish et al. (2013), Utz et al. (2014), and Utz et al. (2015).

While substantial progress has been made as described above, it is quite possible that multiple criteria in portfolio selection is still in its early stages. With criteria beyond expected return and variance causing the efficient *frontier* to become an efficient *surface*, many new questions about the structure of the efficient surface and its relationship to other key quantities in portfolio selection arise. While it is certainly possible to compute individual efficient solutions by means of inserting any additional criterion into the problem as a constraint, this only generates partial information. But to appreciate the full expanse of potentially optimal choices offered by a problem, it is necessary to compute the entire efficient surface. Whereas Merton (1972) has provided a very nice analytical derivation of the bi-criterion efficient frontier of traditional portfolio selection, the purpose of this paper is to provide a similar analytical derivation but of the efficient surface of a tri-criterion portfolio selection problem.

The paper is organized as follows. In Sect. 2 we touch on multiple criteria optimization and summarize many of the main results of the efficient frontier of Merton's model. In Sect. 3 we formulate our tri-criterion model and analytically derive the minimum-variance surface. In Sect. 4 we derive the portion of the minimum-variance surface that is the efficient surface, and in Sect. 5 we provide an illustrative numerical example. Also in Sect. 5, we are able to illustrate the paraboloidic/hyperboloidic nature of the efficient surface by means of several 3D graphs. In Sect. 6 we end the paper with some concluding remarks.

2 Multiple criteria optimization and portfolio selection

We briefly review multiple criteria optimization and portfolio selection models in this section. A multiple objective optimization problem can be formulated as

$$\begin{aligned} &\max \{z_1 = f_1(\mathbf{x})\} \\ &\quad\vdots \\ &\max \{z_k = f_k(\mathbf{x})\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{1}$$

where k is the number of objectives and $S \subset \mathbb{R}^n$ is the feasible region in *decision space*. Because (1) has more than one objective, there is another version of the feasible region, that being $Z \subset \mathbb{R}^k$ in *criterion space*, where $Z = \{\mathbf{z} \mid z_i = f_i(\mathbf{x}),\ \mathbf{x} \in S\}$ with reference to which $\mathbf{z} = (z_1, \ldots, z_k)$ is a *criterion vector*. In criterion space, $\bar{\mathbf{z}} \in Z$ is *nondominated* iff there does not exist an $\mathbf{x} \in S$ such that $f_i(\mathbf{x}) \geq f_i(\bar{\mathbf{x}})$ for all i, with at least one of the inequalities strict. Otherwise, $\bar{\mathbf{z}}$ is *dominated*. The set of all nondominated criterion vectors is called the nondominated set and is designated N. In decision space, $\bar{\mathbf{x}} \in S$ is *efficient* iff its criterion vector $\bar{\mathbf{z}} = (f_1(\bar{\mathbf{x}}), \ldots, f_k(\bar{\mathbf{x}}))$ is nondominated. Otherwise, $\bar{\mathbf{x}}$ is *inefficient*. The set of all efficient points is called the efficient set and is designated E. In the form above, easier said than done, the purpose of (1) is to compute all of N and E for use by the decision maker. More on multiple criteria optimization can be found in Miettinen (1999) and Ehrgott (2005).

One of the oldest, if not the oldest, mechanism for addressing (1) is the e-constraint approach. In this approach, all of the objectives except one are converted to constraints such as in the following

$$\begin{aligned} &\max \{z_1 = f_1(\mathbf{x})\} \\ &\text{s.t.} \quad f_2(\mathbf{x}) = e_2 \\ &\qquad\quad \vdots \\ &\qquad f_k(\mathbf{x}) = e_k \\ &\qquad \mathbf{x} \in S \end{aligned} \tag{2}$$

where the e_i are pre-chosen values of all of the objectives that have been re-modeled as constraints. Typically, (2) is solved many times for different configurations of the e_i.

In Markowitz (1952), his landmark portfolio selection formulation, given in bi-criterion format, is

$$\begin{aligned} &\min\{z_1 = \mathbf{x}^T \Sigma \mathbf{x}\} && \text{variance} \\ &\max\{z_2 = \mu^T \mathbf{x}\} && \text{expected return} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$, n is number of securities in the designated pool, the x_i components of $\mathbf{x}$ are the proportions of capital to be allocated to security i, Σ is the problem's covariance matrix, and μ is the problem's vector of individual security expected returns.

The field of finance calls the N of (3) the "efficient" frontier, but we will henceforth call it the *nondominated frontier*. This is so the terms efficient and inefficient can be reserved only for distinguishing among **x**-vectors (i.e., portfolios) in decision space. Thus, in accordance with multiple criteria optimization, the terms dominated and nondominated will only be used in connection with vectors in criterion space, and the terms efficient and inefficient will only be used in connection with vectors in decision space.

In Merton (1972), Merton provides elegant analytical derivations of many of the quantities and properties of the nondominated and efficient sets of the following portfolio model

$$\begin{aligned} &\min\{z_1 = \mathbf{x}^T \Sigma \mathbf{x}\} \\ &\max\{z_2 = \mu^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{1}^T \mathbf{x} = 1 \end{aligned} \tag{4}$$

where $\mathbf{1}$ is a vector of ones. On one hand, the unlimited nature of the x_i weights is unrealistic, but on the other, the analyticity of the derived results from (4) brings substantial advantages to research and teaching (as seen for example in the text by Huang and Litzenberger, 1988). Because we will be parallelling many of the results of Merton (1972), but with a third criterion included, we will now summarize many of the most important points of Merton so as to serve as a good debarkation point for this paper.

Assuming Σ positive definite, Merton begins with the following e-constraint version of (4)

$$\begin{aligned} &\min\{\tfrac{1}{2}\mathbf{x}^T \Sigma \mathbf{x}\} \\ &\text{s.t.} \quad \mu^T \mathbf{x} = z_2 \\ &\qquad \mathbf{1}^T \mathbf{x} = 1 \end{aligned}$$

where $\mathbf{x}^T \Sigma \mathbf{x}$ is variance and z_2 is an arbitrary value of expected return. Then from the Lagrangian

$$L(\mathbf{x}, g_1, g_2) = \tfrac{1}{2}\mathbf{x}^T \Sigma \mathbf{x} + g_1(z_2 - \mu^T \mathbf{x}) + g_2(1 - \mathbf{1}^T \mathbf{x})$$

where the g_i are the multipliers, the following system

$$\begin{bmatrix} z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \mu^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \mathbf{1} \\ \mathbf{1}^T \Sigma^{-1} \mu & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} a & c \\ c & f \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{5}$$

is obtained. With a, c and f defined as in (5), and the determinant $D = af - cc$ of the 2×2 matrix positive as in Merton (1972), the minimum-variance frontier of (4) is the parabola

$$z_1 = \frac{1}{D}\left(f z_2^2 - 2c z_2 + a\right) \tag{6}$$

Such a parabola is given in Fig. 1. The minimum-variance point $\mathbf{z}^{mv}$ on the parabola and its corresponding portfolio $\mathbf{x}^{mv}$ in decision space are given by

$$\mathbf{z}^{mv} = \left(\frac{1}{f}, \frac{c}{f}\right) \qquad \mathbf{x}^{mv} = \frac{1}{f}\Sigma^{-1}\mathbf{1}$$

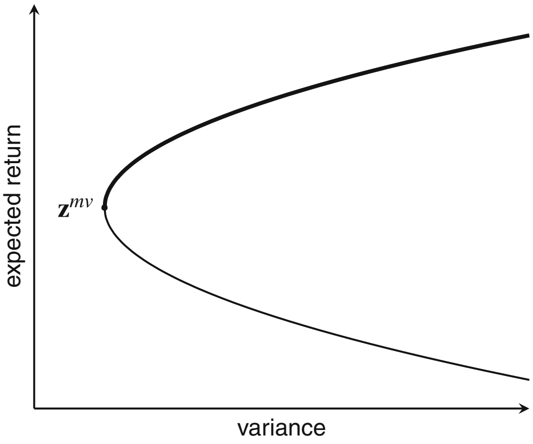


Fig. 1 A minimum-variance frontier plotted in (*variance, expected return*)-space where it is a parabola. The portion of the parabola from $\mathbf{z}^{mv}$ upward is the nondominated frontier in this space

In (*variance, expected return*)-space, the nondominated frontier is the upper part of the parabola starting at $\mathbf{z}^{mv}$. The set of all portfolios that are inverse images of points on the nondominated frontier constitutes the efficient set and is given by

$$\left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{x}^{mv} + \lambda\left(\Sigma^{-1}\mu - \frac{c}{f}\Sigma^{-1}\mathbf{1}\right), \lambda \geq 0\right\} \tag{7}$$

Note that the efficient set of (7) is an unbounded line segment emanating from $\mathbf{x}^{mv}$.

Regarding computational complexity, the most costly part of (6) or (7) is the calculation of the inverse of Σ, which is $O(n^3)$ when using standard Gauss–Jordan elimination. All other operations, the most costly being $n \times n$ by $n \times 1$ matrix multiplications, are at most $O(n^2)$. Thus, the total asymptotic complexity of computing the nondominated frontier and the efficient portfolios in bi-criterion portfolio selection is $O(n^3)$.

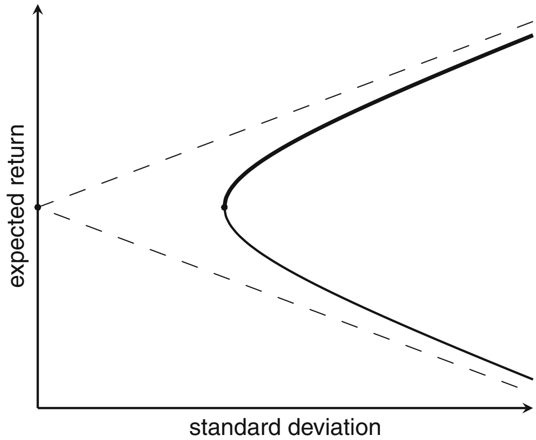
With it common to display the nondominated frontier in (*standard deviation, expected return*)-space, we have Fig. 2. In this figure, because of the change from variance to standard deviation along the horizontal axis, the parabola becomes a hyperbola, with the upper part of the hyperbola starting at the minimum-standard deviation point now showing as the nondominated frontier. The dashed lines in the figure are the asymptotes of the hyperbola given by

$$z_2 = \frac{c}{f} \pm \sqrt{\frac{Dz_1}{f}} \qquad \text{for } z_1 \geq 0$$

where $\sqrt{z_1}$ is standard deviation.

Also covered in Merton (1972) is the 2-mutual-fund theorem. The theorem is as follows. Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be any two portfolios whose criterion vectors are on the minimum-variance (or minimum-standard deviation) frontier, and let $\mathbf{x}$ be any other portfolio (i.e., any vector in $\mathbb{R}^n$ whose components sum to one). Then, the criterion vector of $\mathbf{x}$ is on the minimum-variance (or minimum-standard deviation) frontier iff $\mathbf{x}$ can be formed as a linear combination of $\mathbf{x}^1$ and $\mathbf{x}^2$ whose weights sum to one.

Fig. 2 The same minimum-variance frontier plotted in (*standard deviation, expected return*)-space where it is a hyperbola. The upper part of the hyperbola is the nondominated frontier in this space. The *dashed lines* are the asymptotes of the hyperbola



3 Deriving the minimum-variance surface

Following the literature with regard to the growing interest in additional criteria in portfolio selection, let us add an additional objective to (4) to form the following tri-criterion model

$$\begin{aligned} &\min \{z_1 = \mathbf{x}^T \Sigma \mathbf{x}\} \\ &\max \{z_2 = \mu^T \mathbf{x}\} \\ &\max \{z_3 = \ell^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{1}^T \mathbf{x} = 1 \end{aligned} \tag{8}$$

While there are many candidates for a third criterion as discussed in Steuer et al. (2007), let us motivate our third criterion with portfolio liquidity, an arbitrary choice, for illustrative purposes. Hence, for liquidity, we have the ℓ-vector in (8). As with the solution to model (4), the solution to (8) is all of its nondominated and efficient sets N and E. In analyzing (8), we make the following assumptions.

Assumption 1 Matrix Σ is positive definite.

Assumption 2 Vectors μ, ℓ and $\mathbf{1}$ are linearly independent.

Beginning as in the bi-criterion case, we form the following e-constraint representation of our tri-criterion model

$$\begin{aligned} &\min \{\tfrac{1}{2}\mathbf{x}^T \Sigma \mathbf{x}\} \\ &\text{s.t.} \quad \mu^T \mathbf{x} = z_2 \\ &\qquad \ell^T \mathbf{x} = z_3 \\ &\qquad \mathbf{1}^T \mathbf{x} = 1 \end{aligned} \tag{9}$$

where $\mathbf{x}^T \Sigma \mathbf{x}$ is variance, and z_2 and z_3 are arbitrary values of expected return and liquidity, respectively. The union of all criterion vectors (z_1, z_2, z_3) resulting from the optimal solutions of (9) for all values of z_2 and z_3 is the *minimum-variance surface* of (8). This is seen as a generalization of the minimum-variance frontier of (4). To begin the process of solving (9) for all values of z_2 and z_3, we take the Lagrangian

$$L(\mathbf{x}, g_2, g_3, g_4) = \frac{1}{2}\mathbf{x}^T \Sigma \mathbf{x} + g_2(z_2 - \mu^T \mathbf{x}) + g_3(z_3 - \ell^T \mathbf{x}) + g_4(1 - \mathbf{1}^T \mathbf{x})$$

where the g_i are multipliers. Because $\mathbf{x}^T \Sigma \mathbf{x}$ is strictly convex by virtue of the positive definiteness of Σ, $L(\mathbf{x}, g_2, g_3, g_4)$ is strictly convex and $\mathbf{x}$ is the minimizing solution of (9) iff

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{x}} &= \Sigma \mathbf{x} - g_2 \mu - g_3 \ell - g_4 \mathbf{1} = \mathbf{0} \\
\frac{\partial L}{\partial g_2} &= z_2 - \mu^T \mathbf{x} = 0 \\
\frac{\partial L}{\partial g_3} &= z_3 - \ell^T \mathbf{x} = 0 \\
\frac{\partial L}{\partial g_4} &= 1 - \mathbf{1}^T \mathbf{x} = 0
\end{aligned}$$

Premultiplying the first equation by Σ^{-1} enables us to obtain $\mathbf{x} = (g_2 \Sigma^{-1} \mu + g_3 \Sigma^{-1} \ell + g_4 \Sigma^{-1} \mathbf{1})$. Substituting this into the last three equations of the above yields

$$\begin{aligned}
g_2 \mu^T \Sigma^{-1} \mu + g_3 \mu^T \Sigma^{-1} \ell + g_4 \mu^T \Sigma^{-1} \mathbf{1} &= z_2 \\
g_2 \mu^T \Sigma^{-1} \ell + g_3 \ell^T \Sigma^{-1} \ell + g_4 \ell^T \Sigma^{-1} \mathbf{1} &= z_3 \\
g_2 \mu^T \Sigma^{-1} \mathbf{1} + g_3 \ell^T \Sigma^{-1} \mathbf{1} + g_4 \mathbf{1}^T \Sigma^{-1} \mathbf{1} &= 1
\end{aligned}$$

We introduce notation $\mathbf{C}$ and express the three equations above in matrix form as

$$\mathbf{C} \begin{bmatrix} g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ 1 \end{bmatrix} \tag{10}$$

where

$$\mathbf{C} = \begin{bmatrix} \mu^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \ell & \mu^T \Sigma^{-1} \mathbf{1} \\ \mu^T \Sigma^{-1} \ell & \ell^T \Sigma^{-1} \ell & \ell^T \Sigma^{-1} \mathbf{1} \\ \mu^T \Sigma^{-1} \mathbf{1} & \ell^T \Sigma^{-1} \mathbf{1} & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

We now demonstrate the following property of $\mathbf{C}$.

Lemma 1 *Matrix $\mathbf{C}$ is positive definite.*

Proof Because Σ^{-1} is positive definite, it can function as a covariance matrix. There exists a random vector $\mathbf{v} \in \mathbb{R}^n$ such that the covariance matrix of $\mathbf{v}$ is Σ^{-1}. In this way, $\mathbf{C}$ is the covariance matrix of the random vector $\left[\mu\ \ell\ \mathbf{1}\right]^T \mathbf{v}$ with $\mathbf{C} = \left[\mu\ \ell\ \mathbf{1}\right]^T \Sigma^{-1} \left[\mu\ \ell\ \mathbf{1}\right]$. Thus, for all $\mathbf{y} \in \mathbb{R}^3$ with $\mathbf{y} \neq \mathbf{0}$, we have $\mathbf{y}^T \mathbf{C} \mathbf{y} = \mathbf{y}^T \left[\mu\ \ell\ \mathbf{1}\right]^T \Sigma^{-1} \left[\mu\ \ell\ \mathbf{1}\right] \mathbf{y}$. Define $\mathbf{w} = \left[\mu\ \ell\ \mathbf{1}\right] \mathbf{y}$. By Assumption 2, $\mathbf{w} \neq \mathbf{0}$. Therefore, $\mathbf{y}^T \mathbf{C} \mathbf{y} = \mathbf{w}^T \Sigma^{-1} \mathbf{w} > 0$. Thus, $\mathbf{C}$ is positive definite. □

By the positive definiteness of $\mathbf{C}$, its determinant $|\mathbf{C}| > 0$, and

$$\mathbf{C}^{-1} = \frac{1}{|\mathbf{C}|} \begin{bmatrix} df - ee & ce - bf & be - cd \\ ce - bf & af - cc & bc - ae \\ be - cd & bc - ae & ad - bb \end{bmatrix}$$

Premultiplying (10) by $\mathbf{C}^{-1}$ gives us

$$\begin{bmatrix} g_2 \\ g_3 \\ g_4 \end{bmatrix} = \mathbf{C}^{-1} \begin{bmatrix} z_2 \\ z_3 \\ 1 \end{bmatrix} = \frac{1}{|\mathbf{C}|} \begin{bmatrix} z_2(df-ee) + z_3(ce-bf) + (be-cd) \\ z_2(ce-bf) + z_3(af-cc) + (bc-ae) \\ z_2(be-cd) + z_3(bc-ae) + (ad-bb) \end{bmatrix}_{3\times 1}$$

Substituting the above g_i into the previously derived $\mathbf{x} = (g_2\Sigma^{-1}\mu + g_3\Sigma^{-1}\ell + g_4\Sigma^{-1}\mathbf{1})$ yields

$$\begin{aligned} \mathbf{x} = \frac{1}{|\mathbf{C}|} \big[&(z_2(df-ee) + z_3(ce-bf) + (be-cd))\Sigma^{-1}\mu \\ &+(z_2(ce-bf) + z_3(af-cc) + (bc-ae))\Sigma^{-1}\ell \\ &+(z_2(be-cd) + z_3(bc-ae) + (ad-bb))\Sigma^{-1}\mathbf{1}\big] \end{aligned}$$

or

$$\mathbf{x} = \mathbf{x}^0 + z_2\mathbf{d}^2 + z_3\mathbf{d}^3 \tag{11}$$

where

$$\mathbf{x}^0 = \frac{1}{|\mathbf{C}|}\left[(be-cd)\Sigma^{-1}\mu + (bc-ae)\Sigma^{-1}\ell + (ad-bb)\Sigma^{-1}\mathbf{1}\right] \tag{12}$$

$$\mathbf{d}^2 = \frac{1}{|\mathbf{C}|}\left[(df-ee)\Sigma^{-1}\mu + (ce-bf)\Sigma^{-1}\ell + (be-cd)\Sigma^{-1}\mathbf{1}\right] \tag{13}$$

$$\mathbf{d}^3 = \frac{1}{|\mathbf{C}|}\left[(ce-bf)\Sigma^{-1}\mu + (af-cc)\Sigma^{-1}\ell + (bc-ae)\Sigma^{-1}\mathbf{1}\right] \tag{14}$$

We interpret $\mathbf{x}^0$ as the minimizing solution of (9) when $z_2 = 0$ and $z_3 = 0$. In this way,

$$\left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{x}^0 + z_2\mathbf{d}^2 + z_3\mathbf{d}^3, \quad z_2, z_3 \in \mathbb{R}\right\} \tag{15}$$

is the set of all optimal solutions of (9) for all values of z_2 and z_3, with the three vectors on the right in the set having the following property.

Lemma 2 *Vectors* $\mathbf{x}^0$, $\mathbf{d}^2$ *and* $\mathbf{d}^3$ *are linearly independent.*

Proof Letting $h_0, h_2, h_3 \in \mathbb{R}$, by (12)–(14), we have

$$\begin{aligned} h_0\mathbf{x}^0 + h_2\mathbf{d}^2 + h_3\mathbf{d}^3 = &\frac{h_0}{|\mathbf{C}|}\left[(be-cd)\Sigma^{-1}\mu + (bc-ae)\Sigma^{-1}\ell + (ad-bb)\Sigma^{-1}\mathbf{1}\right] \\ &+ \frac{h_2}{|\mathbf{C}|}\left[(df-ee)\Sigma^{-1}\mu + (ce-bf)\Sigma^{-1}\ell + (be-cd)\Sigma^{-1}\mathbf{1}\right] \\ &+ \frac{h_3}{|\mathbf{C}|}\left[(ce-bf)\Sigma^{-1}\mu + (af-cc)\Sigma^{-1}\ell + (bc-ae)\Sigma^{-1}\mathbf{1}\right] \end{aligned}$$

After rearrangement, we have

$$\begin{aligned} h_0\mathbf{x}^0 + h_2\mathbf{d}^2 + h_3\mathbf{d}^3 = \frac{1}{|\mathbf{C}|}\big\{&\left[(df-ee)h_2 + (ce-bf)h_3 + (be-cd)h_0\right]\Sigma^{-1}\mu \\ &+\left[(ce-bf)h_2 + (af-cc)h_3 + (bc-ae)h_0\right]\Sigma^{-1}\ell \\ &+\left[(be-cd)h_2 + (bc-ae)h_3 + (ad-bb)h_0\right]\Sigma^{-1}\mathbf{1}\big\} \end{aligned}$$

By Assumptions 1 and 2, $\Sigma^{-1}\mu$, $\Sigma^{-1}\ell$ and $\Sigma^{-1}\mathbf{1}$ are linearly independent. Therefore, the necessary and sufficient condition of $h_0\mathbf{x}^0 + h_2\mathbf{d}^2 + h_3\mathbf{d}^3 = \mathbf{0}$ is

$$\begin{aligned}(df - ee)h_2 + (ce - bf)h_3 + (be - cd)h_0 = 0\\ (ce - bf)h_2 + (af - cc)h_3 + (bc - ae)h_0 = 0\\ (be - cd)h_2 + (bc - ae)h_3 + (ad - bb)h_0 = 0\end{aligned}$$

With the above reducing to $\mathbf{C}^{-1}\begin{bmatrix} h_2\\ h_3\\ h_0\end{bmatrix} = \mathbf{0}$, and $\mathbf{C}^{-1}$ nonsingular, the only possibility is that $h_0 = h_2 = h_3 = 0$. Therefore, $\mathbf{x}^0$, $\mathbf{d}^2$ and $\mathbf{d}^3$ are linearly independent. □

With (15) being the set of all portfolios that generate the minimum-variance surface, it is seen that (15) is an affine set, in particular, being a 2-dimensional hyperplane in $\mathbb{R}^n$ offset from the origin by $\mathbf{x}^0$. From this, as an extension of the 2-mutual-fund theorem of bi-criterion portfolio selection mentioned in Sect. 2, we can state the following 3-mutual-fund theorem of tri-criterion portfolio selection.

Theorem 1 *Let $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$ be any three affinely independent*[1] *points from* (15). *Then, any portfolio that generates a point on the minimum-variance surface can be formed by some linear combination of $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$ whose weights sum to one.*

In criterion space, the minimum-variance surface of model (8) is obtained by substituting (11) into $z_1 = \mathbf{x}^T\Sigma\mathbf{x}$ to yield

$$z_1 = \mathbf{d}^{2^T}\Sigma\mathbf{d}^2 z_2^2 + 2\mathbf{d}^{2^T}\Sigma\mathbf{d}^3 z_2 z_3 + \mathbf{d}^{3^T}\Sigma\mathbf{d}^3 z_3^2 + 2\mathbf{d}^{2^T}\Sigma\mathbf{x}^0 z_2 + 2\mathbf{d}^{3^T}\Sigma\mathbf{x}^0 z_3 + \mathbf{x}^{0^T}\Sigma\mathbf{x}^0 \tag{16}$$

where the six coefficients of (16) are specified in detail as

$$\begin{aligned}
\mathbf{d}^{2^T}\Sigma\mathbf{d}^2 &= \frac{1}{|\mathbf{C}|^2}\left(ad^2f^2 - 2ade^2f + ae^4 - b^2df^2 + b^2e^2f + 2bcdef - 2bce^3 - c^2d^2f + c^2de^2\right)\\
\mathbf{d}^{2^T}\Sigma\,\mathbf{d}^3 &= \frac{1}{|\mathbf{C}|^2}\left(-abdf^2 + abe^2f + acdef - ace^3 + b^3f^2 + bc^2df - 3b^2cef + 2bc^2e^2 - c^3de\right)\\
\mathbf{d}^{3^T}\Sigma\,\mathbf{d}^3 &= \frac{1}{|\mathbf{C}|^2}\left(a^2df^2 - a^2e^2f - ab^2f^2 + 2abcef - 2ac^2df + ac^2e^2 + b^2c^2f - 2bc^3e + c^4d\right)\\
\mathbf{d}^{2^T}\Sigma\,\mathbf{x}^0 &= \frac{1}{|\mathbf{C}|^2}\left(abdef - abe^3 - acd^2f + acde^2 - b^3ef + b^2cdf + 2b^2ce^2 - 3bc^2de + c^3d^2\right)\\
\mathbf{d}^{3^T}\Sigma\,\mathbf{x}^0 &= \frac{1}{|\mathbf{C}|^2}\left(-a^2def + a^2e^3 + ab^2ef + abcdf - 3abce^2 + ac^2de - b^3cf + 2b^2c^2e - bc^3d\right)\\
\mathbf{x}^{0^T}\Sigma\,\mathbf{x}^0 &= \frac{1}{|\mathbf{C}|^2}\left(a^2d^2f - a^2de^2 - 2ab^2df + ab^2e^2 + 2abcde - ac^2d^2 - 2b^3ce + b^4f + b^2c^2d\right)
\end{aligned}$$

Let us now comment on the notion of an *elliptic paraboloid*. In (z_1, z_2, z_3)-space, the expression $z_1 = \alpha_2 z_2^2 + \alpha_3 z_3^2$, where $\alpha_2 \geq 0$ and $\alpha_3 \geq 0$, is an elliptic paraboloid in standard form. The paraboloid is *non-degenerate*, if both $\alpha_2 > 0$ and $\alpha_3 > 0$. Otherwise the paraboloid is *degenerate*. For a given value $\zeta > 0$ of z_1, we obtain $\frac{\alpha_2}{\zeta}z_2^2 + \frac{\alpha_3}{\zeta}z_3^2 = 1$ in (z_2, z_3)-space. This is recognized as an ellipsoid.

Theorem 2 *The minimum-variance surface* (16) *of the tri-criterion portfolio problem of* (8) *is a non-degenerate elliptic paraboloid.*

[1] Points $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^m$ are affinely independent if $\mathbf{x}^1 - \mathbf{x}^0, \ldots, \mathbf{x}^m - \mathbf{x}^0$ are linearly independent.

Proof We rewrite (16) as

$$z_1 = \begin{bmatrix} z_2 \ z_3 \ 1 \end{bmatrix} \mathbf{P} \begin{bmatrix} z_2 \\ z_3 \\ 1 \end{bmatrix} \quad \text{where} \quad \mathbf{P} = \begin{bmatrix} \mathbf{d}^{2^T} \Sigma \mathbf{d}^2 & \mathbf{d}^{2^T} \Sigma \mathbf{d}^3 & \mathbf{d}^{2^T} \Sigma \mathbf{x}^0 \\ \mathbf{d}^{2^T} \Sigma \mathbf{d}^3 & \mathbf{d}^{3^T} \Sigma \mathbf{d}^3 & \mathbf{d}^{3^T} \Sigma \mathbf{x}^0 \\ \mathbf{d}^{2^T} \Sigma \mathbf{x}^0 & \mathbf{d}^{3^T} \Sigma \mathbf{x}^0 & \mathbf{x}^{0^T} \Sigma \mathbf{x}^0 \end{bmatrix}.$$

As Σ is a covariance matrix, let $\mathbf{r} \in \mathbb{R}^n$ designate the random vector associated with it. Form a new random vector $\begin{bmatrix} \mathbf{d}^2 \ \mathbf{d}^3 \ \mathbf{x}^0 \end{bmatrix}^T \mathbf{r}$. Let $\mathbf{P}$ be its covariance matrix. For all $\mathbf{y} \in \mathbb{R}^3$ with $\mathbf{y} \neq \mathbf{0}$, $\mathbf{y}^T \mathbf{P} \mathbf{y} = \mathbf{y}^T \begin{bmatrix} \mathbf{d}^2 \ \mathbf{d}^3 \ \mathbf{x}^0 \end{bmatrix}^T \Sigma \begin{bmatrix} \mathbf{d}^2 \ \mathbf{d}^3 \ \mathbf{x}^0 \end{bmatrix} \mathbf{y}$. Let $\mathbf{w} = \begin{bmatrix} \mathbf{d}^2 \ \mathbf{d}^3 \ \mathbf{x}^0 \end{bmatrix} \mathbf{y}$ and $\mathbf{w} \neq \mathbf{0}$ by Lemma 2. Then $\mathbf{y}^T \mathbf{P} \mathbf{y} = \mathbf{w}^T \Sigma \mathbf{w} > 0$, because Σ is positive definite. Therefore, $\mathbf{P}$ is positive definite.

With $\mathbf{P}$ positive definite, all of its eigenvalues v_1, v_2 and v_3 are positive. With $\mathbf{P}$ real and symmetric, there exists a normal matrix $\mathbf{N}$ such that $\mathbf{P} = \mathbf{N}^T \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix} \mathbf{N}$. Hence, the minimum-variance surface (16) is $z_1 = \begin{bmatrix} z_2 \ z_3 \ 1 \end{bmatrix} \mathbf{N}^T \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix} \mathbf{N} \begin{bmatrix} z_2 \\ z_3 \\ 1 \end{bmatrix}$. Let $\mathbf{u} = \mathbf{N} \begin{bmatrix} z_2 \\ z_3 \\ 1 \end{bmatrix}$. Then, $z_1 = \mathbf{u}^T \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix} \mathbf{u} = v_1 u_1^2 + v_2 u_2^2 + v_3 u_3^2$. With $v_1, v_2, v_3 > 0$, after a change of coordinate system, we see the paraboloid as non-degenerate. □

A depiction of a minimum-variance surface is given in Fig. 3. The task of specifying the portion of the paraboloid that is the nondominated set of our tri-criterion model (8) still awaits us.

4 Deriving the nondominated surface

In order to compute the efficient and nondominated sets of model (8), we utilize a weighted-sums approach to form

$$\begin{aligned} \min \ & \left\{ \tfrac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} - \lambda_2 \mu^T \mathbf{x} - \lambda_3 \ell^T \mathbf{x} \right\} \quad \lambda_2, \lambda_3 \geq 0 \\ \text{s.t.} \ & \mathbf{1}^T \mathbf{x} = 1 \end{aligned} \tag{17}$$

whose Langrangian is

$$L(\mathbf{x}, g) = \tfrac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} - \lambda_2 \mu^T \mathbf{x} - \lambda_3 \ell^T \mathbf{x} + g(1 - \mathbf{1}^T \mathbf{x})$$

where g is its multiplier. Because $\mathbf{x}^T \Sigma \mathbf{x}$ is strictly convex, $L(\mathbf{x}, g)$ is strictly convex and $\mathbf{x}$ is the minimizing solution to (17) if and only if it satisfies

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{x}} &= \Sigma \mathbf{x} - \lambda_2 \mu - \lambda_3 \ell - g \mathbf{1} = \mathbf{0} \\ \frac{\partial L}{\partial g} &= 1 - \mathbf{1}^T \mathbf{x} = 0 \end{aligned}$$

Premultiplying the first equation by Σ^{-1} yields $\mathbf{x} = \lambda_2 \Sigma^{-1}\mu + \lambda_3 \Sigma^{-1}\ell + g\Sigma^{-1}\mathbf{1}$. Substituting $\mathbf{x}$ into the second equation produces

$$g = \frac{1}{\mathbf{1}^T \Sigma^{-1}\mathbf{1}} \left(1 - \lambda_2 \mathbf{1}^T \Sigma^{-1}\mu - \lambda_3 \mathbf{1}^T \Sigma^{-1}\ell\right) = \frac{1}{f}\left(1 - \lambda_2 c - \lambda_3 e\right)$$

Noting that $f > 0$, the above is well-defined. Substituting g into $\mathbf{x} = (\lambda_2 \Sigma^{-1}\mu + \lambda_3 \Sigma^{-1}\ell + g\Sigma^{-1}\mathbf{1})$ yields

$$\mathbf{x} = \lambda_2 \Sigma^{-1}\mu + \lambda_3 \Sigma^{-1}\ell + \frac{1}{f}\left(1 - \lambda_2 c - \lambda_3 e\right) \Sigma^{-1}\mathbf{1}$$

or

$$\mathbf{x} = \mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3 \tag{18}$$

where

$$\mathbf{x}^{mv} = \frac{1}{f}\Sigma^{-1}\mathbf{1} \tag{19}$$

$$\Delta^2 = \Sigma^{-1}\mu - \frac{c}{f}\Sigma^{-1}\mathbf{1} \tag{20}$$

$$\Delta^3 = \Sigma^{-1}\ell - \frac{e}{f}\Sigma^{-1}\mathbf{1} \tag{21}$$

Notice that the expression for $\mathbf{x}^{mv}$, the minimum-variance portfolio, is the same for both Merton's model (4) and the tri-criterion model (8) of this paper. The efficient set of (8) can then be expressed as

$$\left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3, \quad \lambda_2, \lambda_3 \geq 0\right\} \tag{22}$$

Lemma 3 *Vectors Δ^2 and Δ^3 are linearly independent.*

Proof For $h_2, h_3 \in \mathbb{R}$ we have

$$\begin{aligned} h_2\Delta^2 + h_3\Delta^3 &= h_2\left(\Sigma^{-1}\mu - \frac{c}{f}\Sigma^{-1}\mathbf{1}\right) + h_3\left(\Sigma^{-1}\ell - \frac{e}{f}\Sigma^{-1}\mathbf{1}\right) \\ &= h_2\Sigma^{-1}\mu + h_3\Sigma^{-1}\ell - \frac{ch_2 + eh_3}{f}\Sigma^{-1}\mathbf{1} \end{aligned} \tag{23}$$

Since $\Sigma^{-1}\mu$, $\Sigma^{-1}\ell$ and $\Sigma^{-1}\mathbf{1}$ are linearly independent, the right-hand side of (23) is zero iff $h_2 = h_3 = 0$, and $-\frac{ch_2+eh_3}{f} = 0$. Since only h_2 and h_3 are needed, Δ^2 and Δ^3 are linearly independent. □

Therefore, generated by Δ^2 and Δ^3, E is a translated 2-dimensional cone. Furthermore, note that the Δ^2 generator of (22) is the same as the single generator of (7). This means that any portfolio efficient in model (4) is efficient in model (8), and this immediately enables us to state the following theorem.

Theorem 3 *The efficient set* (7) *of the bi-criterion model* (4) *is a subset of the efficient set* (22) *of the tri-criterion model* (8).

Thus by adding a linear criterion, the investor's efficient set becomes a superset of its former self. By substituting (18) into model (8) we are able to demonstrate, as a function of $\lambda_2, \lambda_3 \geq 0$, the nondominated set of (8) in the form of the following set of parametric equations

$$\begin{aligned} z_1 &= \left(\mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3\right)^T \Sigma \left(\mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3\right) \\ z_2 &= \mu^T \left(\mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3\right) \\ z_3 &= \ell^T \left(\mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3\right) \end{aligned} \tag{24}$$

Whereas the nondominated set of Merton's bi-criterion model (4) is a portion of the parabolic minimum-variance frontier (6), the nondominated set of the tri-criterion model is a portion of the paraboloidic minimum-variance surface (16).

Comparing the computational complexity of tri-criterion portfolio selection to the computational complexity of bi-criterion portfolio selection, notice that for computing (22) and (24), the calculation of the inverse of Σ is still the most costly part, which is $O(n^3)$ when using standard Gauss–Jordan elimination. As in bi-criterion portfolio selection, all other operations are at most $O(n^2)$. Thus, the total asymptotic complexity of computing the nondominated surface and the efficient portfolios in tri-criterion portfolio selection is $O(n^3)$, which is the same as in bi-criterion portfolio selection. This means that as long as there are no further restrictions on the portfolio weights, adding a third linear criterion to standard portfolio selection comes at virtually no cost.

5 Illustrative numerical example

We now provide a numerical example along with graphs to illustrate the results of this paper. To equip the example, data from the Center for Research in Security Prices (CRSP) were obtained over the period January 2009 to December 2013 on four stocks chosen from the Dow Jones Industrial Average index: American Express (AXP), Disney (DIS), Johnson and Johnson (JNJ), and Coca Cola (KO). Monthly data over the period were downloaded for the covariance matrix Σ and the individual security expected return vector μ of model (8). Also, for the model's liquidity vector ℓ, we downloaded monthly closing bid, closing asked, and closing prices so as to compute as our liquidity measure the negative of each stock's bid-asked spread $\frac{\text{asked price}-\text{bid price}}{\text{closing price}}$. With all of this, we have

$$\mu = \begin{bmatrix} 0.0355 \\ 0.0240 \\ 0.0109 \\ 0.0135 \end{bmatrix} \quad \ell = \begin{bmatrix} -0.0003 \\ -0.0003 \\ -0.0002 \\ -0.0002 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0.0182 & 0.0059 & 0.0016 & 0.0008 \\ 0.0059 & 0.0050 & 0.0014 & 0.0014 \\ 0.0016 & 0.0014 & 0.0018 & 0.0010 \\ 0.0008 & 0.0014 & 0.0010 & 0.0019 \end{bmatrix} \tag{25}$$

Utilizing the μ, ℓ and Σ of (25) in (12)–(14), we obtain

$$\mathbf{x}^0 = \begin{bmatrix} 0.7643 \\ -2.7643 \\ 0.4959 \\ 2.5041 \end{bmatrix} \qquad \mathbf{d}^2 = \begin{bmatrix} 61.8479 \\ -61.8479 \\ -111.0574 \\ 111.0574 \end{bmatrix} \qquad \mathbf{d}^3 = 10^4 \times \begin{bmatrix} 0.7553 \\ -1.7553 \\ -0.6975 \\ 1.6975 \end{bmatrix}$$

With these vectors inserted, in accordance with (15), the following set

$$\left\{\mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} = \mathbf{x}^0 + z_2\mathbf{d}^2 + z_3\mathbf{d}^3, \quad z_2, z_3 \in \mathbb{R}\right\}$$

gives the 2-dimensional hyperplane of portfolios in $\mathbf{x}$-space that generates the minimum-variance surface. And by (16), we have

$$z_1 = 9.5778 \times 10^5 z_2^2 + 1.1396 \times 10^4 z_2 z_3 + 53.5843 z_3^2 + 242.7541 z_2 + 0.9582 z_3 + 0.0198$$

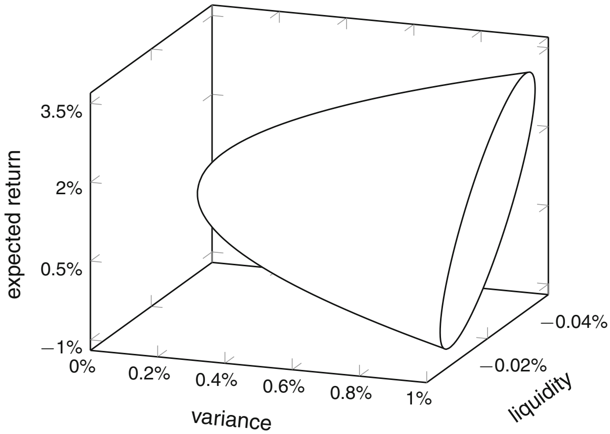


Fig. 3 The portion of the paraboloidic minimum-variance surface of the illustrative numerical example for variance $z_1 \leq .01$

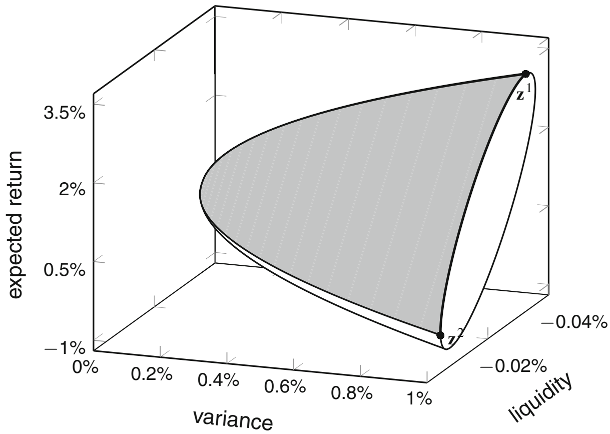


Fig. 4 The portion of the minimum-variance surface that is the nondominated surface for variance $z_1 \leq .01$

as the equation of the elliptic paraboloid that is the minimum-variance surface. Graphing this, the minimum-variance surface is portrayed in Fig. 3.

Now for the nondominated surface. Utilizing the μ, ℓ and Σ of (25) in (19)–(21), we obtain

$$\mathbf{x}^{mv} = \begin{bmatrix} 0.0158 \\ -0.0140 \\ 0.5210 \\ 0.4772 \end{bmatrix} \quad \Delta^2 = \begin{bmatrix} 0.8591 \\ 2.1633 \\ -3.5338 \\ 0.5114 \end{bmatrix} \quad \Delta^3 = \begin{bmatrix} 0.0028 \\ -0.0312 \\ 0.0137 \\ 0.0147 \end{bmatrix}$$

With these vectors inserted, in accordance with (18), the following set

$$\left\{\mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} = \mathbf{x}^{mv} + \lambda_2 \Delta^2 + \lambda_3 \Delta^3, \quad \lambda_2, \lambda_3 \geq 0\right\} \tag{26}$$

gives the 2-dimensional translated cone of efficient portfolios in **x**-space that generates the portion of the minimum-variance surface that is the nondominated surface. By substituting

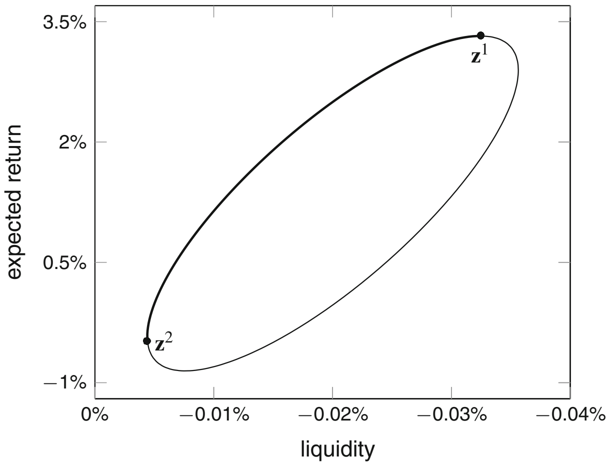


Fig. 5 Cross-section taken at variance $z_1 = .01$ showing the rotated nature of the minimum-variance surface

the vectors of the efficient set (26) into model (8), we obtain the nondominated surface as shown in gray in Fig. 4.

Notice the rightmost oval seen in Fig. 4. It is the cross-section of the minimum-variance surface with constant variance $z_1 = 0.01$. It is shown more directly in Fig. 5. As seen, the major axis of the ellipse is not parallel to either the z_2 or z_3 axis. Thus, the minimum-variance surface is rotated. This is a consequence of the fact that the $\mathbf{d}^{2^T} \Sigma \mathbf{d}^3$ coefficient of the $z_2 z_3$ term in the expression for the elliptic paraboloid above is not equal to 0. The heavier line between $\mathbf{z}^1$ and $\mathbf{z}^2$ inclusive is the portion of the ellipse that is nondominated.

As standard deviation is often more interpretable than variance (as standard deviation is given in the same units as expected return), we now look at our plotting situation in terms of standard deviation. Whereas the parabola of Fig. 1 becomes the hyperbola of Fig. 2 when variance is changed to standard deviation in Merton's bi-criterion model (4), the paraboloid of Fig. 3 becomes the hyperboloid seen in Fig. 6 when variance is changed to standard deviation in our tri-criterion model (8).

Denoting standard deviation by $z_1^* = \sqrt{z_1}$, the hyperboloid in given by

$$z_1^* = \sqrt{9.5778 \times 10^5 z_2^2 + 1.1396 \times 10^4 z_2 z_3 + 53.5843 z_3^2 + 242.7541 z_2 + 0.9582 z_3 + 0.0198}$$

While a hyperbola (as in Merton's model) is surrounded by only 2 asymptotes, a hyperboloid is surrounded by an asymptotic cone. Such an asymptotic cone can be obtained by shifting the vertex of the paraboloid that corresponds to the hyperboloid in the z_1 direction to $z_1 = 0$. Since the vertex of the minimum-variance paraboloid is the minimum-variance point, this involves a shift of $z_1^{mv} = \frac{1}{f} = 1.4206 \times 10^{-3}$ in our example. Thus, the asymptotic cone is given by

$$\begin{aligned} &\sqrt{(z_1^*)^2 + 1.4206 \times 10^{-3}} \\ &\quad = \sqrt{9.5778 \times 10^5 z_2^2 + 1.1396 \times 10^4 z_2 z_3 + 53.5843 z_3^2 + 242.7541 z_2 + 0.9582 z_3 + 0.0198} \end{aligned}$$

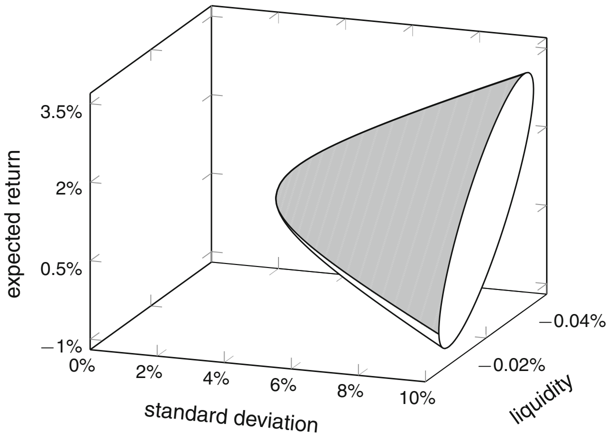


Fig. 6 The portions of the hyperboloidic minimum-standard deviation surface and nondominated surface of the illustrative numerical example for standard deviation $\sqrt{z_1} \le .10$

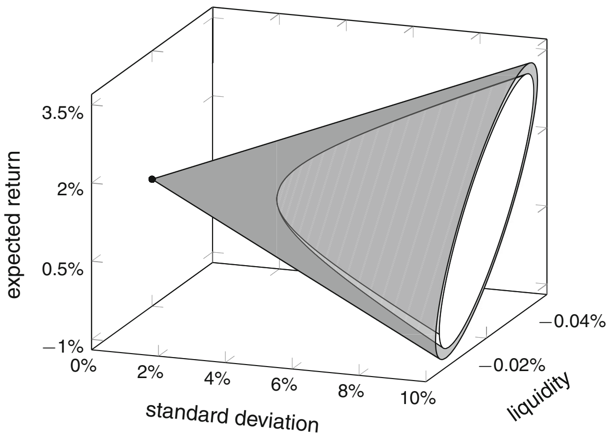


Fig. 7 Figure 6 with enclosing asymptotic cone for standard deviation $\sqrt{z_1} \le .10$

or equivalently

$$\begin{aligned}(z_1^*)^2 = {} & 9.5778 \times 10^5 z_2^2 + 1.1396 \times 10^4 z_2 z_3 + 53.5843 z_3^2 \\ & + 242.7541 z_2 + 0.9582 z_3 + 0.0184\end{aligned}$$

How the asymptotic cone encloses the hyperboloidic minimum-standard deviation surface and nondominated surface is shown in Fig. 7. The dot in the *liquidity, expected return* plane is the origin of the cone. This ends our illustrative example.

6 Concluding remarks

The advantage of the analytical derivation of this paper for a tri-criterion mean-variance portfolio selection problem whose third criterion is linear is that any defined point on the nondominated surface as well as a full mathematical specification of the nondominated surface can be computed directly by means of a formula. That is, no mathematical programming is required. With the most difficult part of any of the formulas being the inverse of the covariance matrix, and with Matlab able to compute the inverse of, say, a 500×500 covariance matrix in only a few hundredths of a second, this means that any particular portfolio result can generally be obtained analytically in nearly unnoticeable time on that platform. Of course, as in Merton (1972), but for the tri-criterion situation, this is only for problems that have a single equality constraint and an invertible covariance matrix. If the covariance matrix is not invertible or the tri-criterion problem has an inequality constraint, then mathematical programming is required, and any desired point on the nondominated surface as well as a full mathematical specification of the nondominated surface will take much more time. For example, using, as of this writing, the fastest tri-criterion algorithm known to exist, that is, CIOS from Hirschberger et al. (2013), a tri-criterion problem with a 500×500 covariance matrix would take in contrast over six seconds. Thus one would want to use the analytical derivation whenever possible. Other advantages of the analytical approach are that by means of the formulas one can see more clearly the mathematics of the relationships among the various portfolio quantities, and that the analytical derivation of this paper allows the teaching of tri-criterion portfolio selection without requiring students to know optimization, which can be of great convenience in many pedagogical situations.

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References

Aouni, B., Colapinto, C., & La Torre, D. (2014). Financial portfolio management through the goal programming model: Current state-of-the-art. *European Journal of Operational Research*, *234*(2), 536–545.

Ballestero, E., Bravo, M., Pérez-Gladish, B., Arenas-Parra, M., & Plà-Santamaria, D. (2012). Socially responsible investment: A multicriteria approach to portfolio selection combining ethical and financial objectives. *European Journal of Operational Research*, *216*(2), 487–494.

Bana e Costa, C. A., & Soares, J. O. (2001). Multicriteria approaches for portfolio selection: An overview. *Review of Financial Markets*, *4*(1), 19–26.

Ben Abdelaziz, F., Aouni, B., & El Fayedh, R. (2007). Multi-objective stochastic programming for portfolio selection. *European Journal of Operational Research*, *177*(3), 1811–1823.

Bilbao-Terol, A., Arenas-Parra, M., Cañal-Fernández, V., & Bilbao-Terol, C. (2013). Selection of socially responsible portfolios using hedonic prices. *Journal of Business Ethics*, *115*(3), 515–529.

Cabello, J. M., Ruiz, F., Pérez-Gladish, B., & Méndez-Rodriguez, P. (2014). Synthetic indicators of mutual funds' environmental responsibility: An application of the reference point method. *European Journal of Operational Research*, *236*(1), 313–325.

Calvo, C., Ivorra, C., & Liern, V. (2014). Fuzzy portfolio selection with non-financial goals: Exploring the efficient frontier. *Annals of Operations Research*. doi:10.1007/s10479-014-1561-2.

Chow, G. (1995). Portfolio selection based on return, risk, and relative performance. *Financial Analysts Journal*, *51*(2), 54–60.

Dorfleitner, G., Leidl, M., & Reeder, J. (2012). Theory of social returns in portfolio choice with application to microfinance. *Journal of Asset Management*, *13*(6), 384–400.

Ehrgott, M. (2005). *Multicriteria optimization* (2nd ed.). Berlin: Springer.

Ehrgott, M., Klamroth, K., & Schwehm, C. (2004). An MCDM approach to portfolio optimization. *European Journal of Operational Research*, *155*(3), 752–770.

Feuersänger, C. (2014). *Manual for package PGFPLOTS, version 1.11*, Universität Bonn.

Guerard, J. B., & Mark, A. (2004). Data show importance of company R&D in picking stocks. *Pensions & Investments*, *32*(25), 30–31.

Hallerbach, W. G., Ning, H., Soppe, A., & Spronk, J. (2004). A framework for managing a portfolio of socially responsible investments. *European Journal of Operational Research*, *153*(2), 517–529.

Hirschberger, M., Steuer, R. E., Utz, S., Wimmer, M., & Qi, Y. (2013). Computing the nondominated surface in tri-criterion portfolio selection. *Operations Research*, *61*(1), 169–183.

Huang, C., & Litzenberger, R. H. (1988). *Foundations for financial economics*. Englewood Cliffs, NJ: Prentice Hall.

Konno, H., & Suzuki, K. I. (1995). A mean-variance-skewness portfolio optimization model. *Journal of the Operations Research Society of Japan*, *38*(2), 173–187.

Lee, S. M. (1972). *Goal programming for decision analysis*. Philadelphia: Auerbach Publishers.

Lo, A. W., Petrov, C., & Wierzbicki, M. (2003). It's 11pm—do you know where your liquidity is? The mean-variance-liquidity frontier. *Journal of Investment Management*, *1*(1), 55–93.

Markowitz, H. M. (1952). Portfolio selection. *Journal of Finance*, *7*(1), 77–91.

Merton, R. C. (1972). An analytical derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis*, *7*(4), 1851–1872.

Miettinen, K. (1999). *Nonlinear multiobjective optimization*. Boston: Kluwer.

Pérez-Gladish, B., Méndez-Rodriguez, P., M'Zali, B., & Lang, P. (2013). Mutual funds efficiency measurement under financial and social responsibility criteria. *Journal of Multi-Criteria Decision Analysis*, *20*(3–4), 109–125.

Speranza, M. G. (1996). A heuristic algorithm for a portfolio optimization model applied to the Milan stock market. *Computers & Operations Research*, *23*(5), 431–441.

Spronk, J., & Hallerbach, W. G. (1997). Financial modelling: Where to go? With an illustration for portfolio management. *European Journal of Operational Research*, *99*(1), 113–127.

Spronk, J., & Zambruno, G. M. (1981). A multiple-criteria approach to portfolio selection. In Göppl, H., Henn, R. (Eds.) *Geld, Banken und Versicherungen, Band 1, Athenum*, pp. 451–459.

Steuer, R. E., & Na, P. (2003). Multiple criteria decision making combined with finance: A categorized bibliography. *European Journal of Operational Research*, *150*(3), 496–515.

Steuer, R. E., Qi, Y., & Hirschberger, M. (2007). Suitable-portfolio investors, nondominated frontier sensitivity, and the effect of multiple objectives on standard portfolio selection. *Annals of Operations Research*, *152*(1), 297–317.

Stone, B. K. (1973). A linear programming formulation of the general portfolio selection problem. *Journal of Financial and Quantitative Analysis*, *8*(4), 621–636.

Utz, S., Wimmer, M., Hirschberger, M., & Steuer, R. E. (2014). Tri-criterion inverse portfolio optimization with application to socially responsible mutual funds. *European Journal of Operational Research*, *234*(2), 491–498.

Utz, S., Wimmer, M., & Steuer, R. E. (2015). Tri-criterion modeling for creating more-sustainable mutual funds. *European Journal of Operational Research*. doi:10.1016/j.ejor.2015.04.035.

Xidonas, P., Mavrotas, G., Krintas, T., Psarras, J., & Zopounidis, C. (2012). *Multicriteria portfolio management*. Berlin: Springer.

Ziemba, W. (2006). *Personal communication at 21st European Conference on Operational Research*, Reykjavik, Iceland, July 3.