APPENDIX

Should the Private Sector Provide Public Capital?

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Appendix A

The appendix describes the optimization problem and the equilibrium dynamics for the private agent under the three regimes of public capital provision.

A1. Private Provision with a Government Subsidy

The Hamiltonian function for the private agent in this regime can be expressed as

$$H^L = \frac{1}{\gamma}\left(C^L\right)^{\gamma} e^{-\beta t} + \lambda^L e^{-\beta t}\left[\dot{N}^L - r\left(\frac{N^L}{K_T^L}\right)N^L - C^L - \Psi\left(\frac{I^L}{K^L}\right) - \left(1-s^L\right)\Omega\left(\frac{G^L}{K_G^L}\right) + \left(1-\tau_Y^L\right)Y^L\right]$$

$$+q_1^L e^{-\beta t}[I^L - \delta_K K^L - \dot{K}^L] + q_2^L e^{-\beta t}[G^L - \delta_G K_G^L - \dot{K}_G^L] \quad \text{(A1.1)}$$

where, $Y^L = A\left[a\left(K_G^L\right)^{-\rho} + b\left(K^L\right)^{-\rho} + \eta\left(\bar{K}_G^L\right)^{-\rho}\right]^{-1/\rho}$.

Recalling the stationary variables $z^L = K_G^L / K^L$, $c^L = C^L / K^L$, $n^L = N^L / K^L$, and $y^L = Y^L / K^L$, the above optimization exercise leads to the following equilibrium dynamics (where we have set $K_G = \bar{K}_G$ in equilibrium):

$$\frac{\dot{z}^L}{z^L} \equiv \frac{\dot{K}_G^L}{K_G^L} - \frac{\dot{K}^L}{K^L} = \left[\frac{q_G^L - \left(1-s^L\right)}{\left(1-s^L\right)h_2} - \delta_G\right] - \left[\frac{q_K^L - 1}{h_1} - \delta_K\right] \quad \text{(A1.2a)}$$

$$\frac{\dot{n}^L}{n^L} \equiv \frac{\dot{N}^L}{N^L} - \frac{\dot{K}^L}{K^L} = r\left(n^L, z^L\right) + \frac{1}{n^L}\left[c^L + \frac{\left(q_K^L\right)^2 - 1}{2h_1} + \frac{\left\{\left(q_G^L\right)^2 - \left(1-s^L\right)^2\right\} z^L}{2\left(1-s^L\right)^2 h_2} - y^L\right] - \left[\frac{q_K^L - 1}{h_1} - \delta_K\right] \quad \text{(A1.2b)}$$

$$\frac{\dot{c}^L}{c^L} \equiv \frac{\dot{C}^L}{C^L} - \frac{\dot{K}^L}{K^L} = \left[\frac{r\left(n^L, z^L\right) - \beta}{\left(1-\gamma\right)}\right] - \left[\frac{q_K^L - 1}{h_1} - \delta_K\right] \quad \text{(A1.2c)}$$

$$\frac{\dot{q}_K^L}{q_K^L} + \frac{\left(1-\tau_Y^L\right) bA^{-\rho}\left(y^L\right)^{1+\rho}}{q_K^L} + \frac{\left(q_K^L - 1\right)^2}{2h_1 q_K^L} - \delta_K = r\left(n^L, z^L\right) \quad \text{(A1.2d)}$$

$$\frac{\dot{q}_G^L}{q_G^L} + \frac{\left(1-\tau_Y^d\right)\left(1-b-\eta\right) A^{-\rho}\left(y^L / z^L\right)^{1+\rho}}{q_G^L} + \frac{\left[q_G^L - \left(1-s^L\right)\right]^2}{2\left(1-s^L\right)h_2 q_G^L} - \delta_G = r\left(n^L, z^L\right) \quad \text{(A1.2e)}$$

The fraction of output devoted to public investment, g^L $(=G^L/Y^L)$ is given by

$$g^L(t)=\left[\frac{q_G^L-(1-s^L)}{(1-s^L)h_2}\right]\left(\frac{z^L}{y^L}\right) \tag{A1.2f}$$

Differentiating (A1.2f) with respect to time, we can describe the dynamic evolution of $g^L(t)$:

$$\dot{g}^L(t)=\frac{1}{(y^L/z^L)}\left[\frac{\dot{q}_G^L}{(1-s^L)h_2}+g^L(t)\left\{y^L-\left(\partial y^L/\partial z^L\right)\right\}\frac{\dot{z}^L}{z^L}\right] \tag{A1.2g}$$

The steady-state equilibrium is attained when $\dot{z}^L=\dot{n}^L=\dot{c}^L=\dot{q}_K^L=\dot{q}_G^L=0$, and is characterized by sustained balanced growth:

$$\tilde{\phi}\equiv\frac{\dot{K}}{K}=\frac{\dot{K}_G}{K_G}=\frac{\dot{N}}{N}=\frac{\dot{C}}{C}=\frac{\dot{Y}}{Y}$$

Further, from (A1.2g) we see that these conditions imply that $\dot{g}^L=0$. Imposing these conditions in (A1.2a)-(A1.2g), along with the government's balanced budget condition (7), leads to the steady-state conditions (13a)-(13f) in the text.

The linearized dynamics corresponding to the steady-state equilibrium (13a)-(13f) can be described as

$$\underline{\dot{\mathrm{X}}}^L=\Delta^L\left(\underline{\mathrm{X}}^L-\underline{\tilde{\mathrm{X}}}^L\right) \tag{A1.3}$$

where $\underline{\mathrm{X}}^{L'}=\left(z^L,n^L,c^L,q_K^L,q_G^L\right)$, $\underline{\tilde{\mathrm{X}}}^{L'}=\left(\tilde{z}^L,\tilde{n}^L,\tilde{c}^L,\tilde{q}_K^L,\tilde{q}_G^L\right)$, and $\Delta^L\equiv\left[a_{ij}\right]_{5\mathrm{x}5}$, $i,j=1,\ldots,5$ represents the 5x5 coefficient matrix of the linearized system.[1] For the analytics, we assume the following functional form for the upward-sloping supply curve of debt:

$$r\left(n^i,z^i\right)=r^*+e^{\alpha n^i/\left(1+z^i\right)}-1,\ \alpha\geq 0,\ i=L,R,D \tag{A1.4}$$

The linearized matrix can be described as follows:

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[1] The determinant of the coefficient matrix of (A1.3) can be shown to be positive under the condition that $r(\tilde{n}^L,\tilde{z}^L)>\tilde{\phi}$. Since (A1.2a-A1.2e) is a fifth-order system, a positive determinant implies that there could be 1, 3 or 5 positive (unstable) roots. In order to yield a saddlepoint-stable solution, we require that there be three unstable roots, to match the three "jump" variables (c^L,q_K^L,q_G^L). Our numerical simulations yield saddlepoint-stable behavior for all plausible ranges of parameters, with three positive (unstable) and two negative (stable) roots.

$$a_{11}=0;\ a_{12}=0;\ a_{13}=0;\ a_{14}=-\tilde{z}^L/h_1;\ a_{15}=\tilde{z}^L\Big/\left(1-\tilde{s}^L\right)h_2$$

$$a_{21}=-\frac{\alpha(\tilde{n}^L)^2 e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}}{(1+\tilde{z}^L)^2}+\frac{\left[\left(\tilde{q}_G^L\right)^2-(1-\tilde{s}^L)^2\right]}{2(1-\tilde{s}^L)^2h_2}-\frac{\partial\tilde{y}^L}{\partial\tilde{z}^L},\ a_{22}=\frac{\alpha\tilde{n}^L e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}}{(1+\tilde{z}^L)}+\tilde{r}^L-\tilde{\phi}^L,\ a_{23}=1,$$

$$a_{24}=\frac{\tilde{q}_K^L-\tilde{n}^L}{h_1},\ a_{25}=\frac{\tilde{z}^L\tilde{q}_G^L}{(1-\tilde{s}^L)^2h_2}.$$

$$a_{31}=-\frac{\alpha\tilde{c}^L\tilde{n}^L e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}}{(1-\gamma)(1+\tilde{z}^L)^2},\ a_{32}=\frac{\alpha(1+\tau_y^L)e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}\tilde{c}^L}{(1-\gamma)(1+\tilde{z}^L)},\ a_{33}=0,\ a_{34}=-\tilde{c}^L/h_1,\ a_{35}=0.$$

$$a_{41}=-\frac{\alpha\tilde{q}_K^L\tilde{n}^L e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}}{(1+\tilde{z}^L)^2}-bA^{-\rho}(1+\rho)(1-\tau_y^L)(\tilde{y}^L)^{\rho}\left(\frac{\partial\tilde{y}^L}{\partial\tilde{z}^L}\right),\ a_{42}=\frac{\alpha(1+\tau_y^L)\tilde{q}_K^L e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}}{(1+\tilde{z}^L)},\ a_{43}=0,$$

$$a_{44}=(1+\tau_y^L)\tilde{r}^L-\tilde{\phi}^L,\ a_{45}=0.$$

$$a_{51}=-\frac{\alpha\tilde{q}_K^L\tilde{n}^L e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}}{(1+\tilde{z}^L)^2}-(1-b-\eta)A^{-\rho}(1+\rho)(1-\tau_y^L)\left(\frac{\tilde{y}^L}{\tilde{z}^L}\right)^{\rho}\left(\frac{\partial\tilde{y}^L/\partial\tilde{z}^L}{\tilde{z}^L}-\frac{\tilde{y}^L}{\left(\tilde{z}^L\right)^2}\right),$$

$$a_{52}=\frac{\alpha(1+\tau_y^L)e^{\alpha\tilde{n}^L/(1+\tilde{z}^L)}\tilde{q}_G^L}{(1+\tilde{z}^L)},\ a_{53}=0,\ a_{54}=0,\ a_{55}=(1+\tau_y^L)\tilde{r}^L-\tilde{\phi}^L,$$

where $\dfrac{\partial\tilde{y}^L}{\partial\tilde{z}^L}=(1-b)A^{-\rho}\left(\dfrac{\tilde{y}^L}{\tilde{z}^L}\right)^{1+\rho}$.

A2. Private Provision with Government Regulation

The Hamiltonian for the private agent in this regime is given by

$$H^R=\frac{1}{\gamma}\left(C^R\right)^{\gamma}e^{-\beta t}+\lambda^R e^{-\beta t}\left[\dot{N}^R-r\left(\frac{N^R}{K_T^R}\right)N^R-C^R-\Psi\left(\frac{I^R}{K^R}\right)-\Omega\left(\frac{G^R}{K_G^R}\right)+Y^R\right]$$

$$+q_1^R e^{-\beta t}[I^R-\delta_K K^R-\dot{K}^R]+q_2^R e^{-\beta t}[G^R-\delta_G K_G^R-\dot{K}_G^R]+ve^{-\beta t}[g^R Y^R-G^R] \qquad \text{(A2.1)}$$

The equilibrium dynamics for this regime can be expressed as

$$\frac{\dot{z}^R}{z^R}\equiv\frac{\dot{K}_G^R}{K_G^R}-\frac{\dot{K}^R}{K^R}=\left[\frac{q_G^R-v^R-1}{h_2}-\delta_G\right]-\left[\frac{q_K^R-1}{h_1}-\delta_K\right] \qquad \text{(A2.2a)}$$

$$\frac{\dot{n}^R}{n^R} \equiv \frac{\dot{N}^R}{N^R} - \frac{\dot{K}^R}{K^R} = r\left(n^R, z^R\right) + \frac{1}{n^R}\left[c^R + \frac{\left(q_K^R\right)^2 - 1}{2h_1} + \frac{\left\{\left(q_G^R - v^R\right)^2 - 1\right\} z^R}{2h_2} - y^R\right] - \left[\frac{q_K^R - 1}{h_1} - \delta_K\right] \qquad \text{(A2.2b)}$$

$$\frac{\dot{c}^R}{c^R} \equiv \frac{\dot{C}^R}{C^R} - \frac{\dot{K}^R}{K^R} = \left[\frac{r\left(n^R, z^R\right) - \beta}{(1-\gamma)}\right] - \left[\frac{q_K^R - 1}{h_1} - \delta_K\right] \qquad \text{(A2.2c)}$$

$$\frac{\dot{q}_K^R}{q_K^R} + \frac{\left(1 + v^R g^R\right) bA^{-\rho}\left(y^R\right)^{1+\rho}}{q_K^R} + \frac{\left(q_K^R - 1\right)^2}{2h_1 q_K^R} - \delta_K = r\left(n^R, z^R\right) \qquad \text{(A2.2d)}$$

$$\frac{\dot{q}_G^R}{q_G^R} + \frac{\left(1 + v^R g^R\right)\left(1 - b - \eta\right) A^{-\rho}\left(y^R / z^R\right)^{1+\rho}}{q_G^R} + \frac{\left[q_G^R - v^R - 1\right]^2}{2h_2 q_G^R} - \delta_G = r\left(n^R, z^R\right) \qquad \text{(A2.2e)}$$

where, $q_K^R = q_1^R / \lambda^R$, $q_G^R = q_2^R / \lambda^R$, $v^R = v / \lambda^R$.

The evolution of the resource cost of regulation (v^R) can be derived from the first-order condition for investment in public capital (G^R):

$$1 + h_2 g^R \left(\frac{y^R}{z^R}\right) = q_G^R - v^R \qquad \text{(A2.3)}$$

Setting $\dot{z}^R = \dot{n}^R = \dot{c}^R = \dot{q}_K^R = \dot{q}_G^R = 0$ in (A2.2a)-(A2.2e) and noting (A2.3) leads to the steady-state conditions (15a)-(15f) in the text. The linearized dynamics corresponding to this steady state can be described as

$$\underline{\dot{X}}^R = \Delta^R\left(\underline{X}^R - \underline{\tilde{X}}^R\right) \qquad \text{(A2.4)}$$

where $\underline{X}^{R'} = \left(z^R, n^R, c^R, q_K^R, q_G^R\right)$, $\underline{\tilde{X}}^{R'} = \left(\tilde{z}^R, \tilde{n}^R, \tilde{c}^R, \tilde{q}_K^R, \tilde{q}_G^R\right)$, and $\Delta^R \equiv \left[b_{ij}\right]_{5\text{x}5}$, $i, j = 1,...,5$ represents the 5x5 coefficient matrix of the linearized system, where

$$b_{11} = g^R\left[\frac{\partial \tilde{y}^R}{\partial \tilde{z}^R} - \frac{\tilde{y}^R}{\tilde{z}^R}\right],\ b_{12} = 0,\ b_{13} = 0,\ b_{14} = -\tilde{z}^R / h_1,\ b_{15} = 0.$$

$$b_{21} = -\frac{\alpha(\tilde{n}^R)^2 e^{\alpha \tilde{n}^R/(1+\tilde{z}^R)}}{(1+\tilde{z}^R)^2} + \frac{\left[\left(\tilde{q}_G^R - \tilde{v}^R\right)^2 - 1\right]}{2h_2} + g^R(\tilde{q}_G^R - \tilde{v}^R)\left[\frac{\partial \tilde{y}^R}{\partial \tilde{z}^R} - \frac{\tilde{y}^R}{\tilde{z}^R}\right] - \frac{\partial \tilde{y}^R}{\partial \tilde{z}^R},$$

$$b_{22}=\frac{\alpha\tilde{n}^R e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}}{(1+\tilde{z}^R)}+\tilde{r}^R-\tilde{\phi}^R,\ b_{23}=1,\ b_{24}=\frac{\tilde{q}_K^R-\tilde{n}^R}{h_1},\ b_{25}=0.$$

$$b_{31}=-\frac{\alpha\tilde{c}^R\tilde{n}^R e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}}{(1-\gamma)(1+\tilde{z}^R)^2},\ b_{32}=\frac{\alpha e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}\tilde{c}^R}{(1-\gamma)(1+\tilde{z}^R)},\ b_{33}=0,\ b_{34}=-\tilde{c}^R/h_1,\ b_{35}=0.$$

$$b_{41}=-\frac{\alpha\tilde{q}_K^R\tilde{n}^R e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}}{(1+\tilde{z}^R)^2}-bA^{-\rho}(1+\rho)(\tilde{y}^L)^{\rho}\left[h_2\left(g^R\right)^2\left(\frac{\tilde{y}^R}{\tilde{z}^R}\right)\left(\frac{\partial\tilde{y}^R}{\partial\tilde{z}^R}-\frac{\tilde{y}^R}{\tilde{z}^R}\right)-(1+\rho)(1+\tilde{v}^R g^R)\left(\frac{\partial\tilde{y}^R}{\partial\tilde{z}^R}\right)\right],$$

$$b_{42}=\frac{\alpha\tilde{q}_K^R e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}}{(1+\tilde{z}^R)},\ b_{43}=0,\ b_{44}=\tilde{r}^R-\tilde{\phi}^R,\ b_{45}=-bg^R A^{-\rho}\left(\tilde{y}^R\right)^{1+\rho}.$$

$$b_{51}=-\frac{\alpha\tilde{q}_K^R\tilde{n}^R e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}}{(1+\tilde{z}^R)^2}-(1-b-\eta)A^{-\rho}\left(\frac{\tilde{y}^R}{\tilde{z}^R}\right)^{\rho}\left[\frac{\partial\tilde{y}^R/\partial\tilde{z}^R}{\tilde{z}^R}-\frac{\tilde{y}^R}{\left(\tilde{z}^R\right)^2}\right]\left[(1+\tilde{v}^R g^R)(1+\rho)-h_2 g^R\left(\frac{\tilde{y}^R}{\tilde{z}^R}\right)\right]$$

$$-(\tilde{q}_G^R-\tilde{v}^R-1)\left(\frac{g^R}{\tilde{z}^R}\right)\left[\frac{\partial\tilde{y}^R/\partial\tilde{z}^R}{\tilde{z}^R}-\frac{\tilde{y}^R}{\left(\tilde{z}^R\right)^2}\right],$$

$$b_{52}=\frac{\alpha e^{\alpha\tilde{n}^R/(1+\tilde{z}^R)}\tilde{q}_G^R}{(1+\tilde{z}^R)},\ b_{53}=0,\ b_{54}=0,\ b_{55}=\tilde{r}^L-(1-b-\eta)A^{-\rho}g^R\left(\frac{\tilde{y}^R}{\tilde{z}^R}\right)^{1+\rho}+\delta_G,$$

where $\dfrac{\partial\tilde{y}^R}{\partial\tilde{z}^R}=(1-b)A^{-\rho}\left(\dfrac{\tilde{y}^R}{\tilde{z}^R}\right)^{1+\rho}$.

A3. Direct Government Provision

The relevant Hamiltonian for the private agent's optimization problem is given by

$$H^D=\frac{1}{\gamma}\left(C^D\right)^{\gamma}e^{-\beta t}+\lambda^D e^{-\beta t}\left[\dot{N}^D-r\left(N^D/K_T^D\right)N^D-C^D-\Psi\left(I^D/K^D\right)+\left(1-\tau_Y^D\right)Y^D\right]$$
$$+q_K^D e^{-\beta t}[I^D-\delta_K K^D-\dot{K}^D] \qquad \text{(A3.1)}$$

where, $Y^R=A\left[(1-b)\left(K_G^D\right)^{-\rho}+b\left(K^D\right)^{-\rho}\right]^{-1/\rho}$.

The equilibrium dynamics for this regime can be expressed as

$$\frac{\dot{z}^D}{z^D} \equiv \frac{\dot{K}_G^D}{K_G^D} - \frac{\dot{K}^D}{K^D} = \left[g^D\left(\frac{y^D}{z^D}\right) - \delta_G \right] - \left[\frac{q_K^D - 1}{h_1} - \delta_K \right] \tag{A3.2a}$$

$$\frac{\dot{n}^D}{n^D} \equiv \frac{\dot{N}^D}{N^D} - \frac{\dot{K}^D}{K^D} = r\left(n^D, z^D\right) + \frac{1}{n^D}\left[c^D + \frac{\left(q_K^D\right)^2 - 1}{2h_1} + \frac{h_2}{2}\left(g^D\right)^2\left(\frac{(y^D)^2}{2}\right) - \left(1 - g^D\right)y^D \right]$$

$$- \left[\frac{q_K^D - 1}{h_1} - \delta_K \right] \tag{A3.2b}$$

$$\frac{\dot{c}^D}{c^D} \equiv \frac{\dot{C}^D}{C^D} - \frac{\dot{K}^D}{K^D} = \left[\frac{r\left(n^D, z^D\right) - \beta}{(1-\gamma)} \right] - \left[\frac{q_K^D - 1}{h_1} - \delta_K \right] \tag{A3.2c}$$

$$\frac{\dot{q}_K^D}{q_K^D} + \frac{\left(1 - \tau_Y^D\right) bA^{-\rho}\left(y^D\right)^{1+\rho}}{q_K^D} + \frac{\left(q_K^D - 1\right)^2}{2h_1 q_K^D} - \delta_K = r\left(n^D, z^D\right) \tag{A3.2d}$$

Setting $\dot{z}^D = \dot{n}^D = \dot{c}^D = \dot{q}_K^D = 0$ in (A2.2a)-(A3.2d) leads to the steady-state conditions (16a)-(16d) in the text. The linearized dynamics corresponding to this steady state can be described as

$$\dot{\underline{\mathrm{X}}}^D = \Delta^D\left(\underline{\mathrm{X}}^D - \tilde{\underline{\mathrm{X}}}^D\right) \tag{A3.3}$$

where $\underline{\mathrm{X}}^{D'} = \left(z^D, n^D, c^D, q_K^D\right)$, $\tilde{\underline{\mathrm{X}}}^{D'} = \left(\tilde{z}^D, \tilde{n}^D, \tilde{c}^D, \tilde{q}_K^D\right)$, and $\Delta^D \equiv \left[c_{ij}\right]_{4x4}$, $i, j = 1,...,4$ represents the 4x4 coefficient matrix of the linearized system, where

$$c_{11} = g^D\left[\frac{\partial \tilde{y}^D}{\partial \tilde{z}^D} - \frac{\tilde{y}^D}{\tilde{z}^D}\right],\ c_{12} = 0,\ c_{13} = 0,\ c_{14} = -\tilde{z}^D / h_1.$$

$$c_{21} = -\frac{\alpha(\tilde{n}^D)^2 e^{\alpha\tilde{n}^D/(1+\tilde{z}^D)}}{(1+\tilde{z}^D)^2} - (1-g^D)\left(\frac{\partial\tilde{y}^D}{\partial\tilde{z}^D}\right) + \frac{h_2}{2}\left(g^D\right)^2\left[2\left(\frac{\tilde{y}^D}{\tilde{z}^D}\right)\left(\frac{\partial\tilde{y}^D}{\partial\tilde{z}^D}\right) - \left(\frac{\tilde{y}^D}{\tilde{z}^D}\right)^2\right],$$

$$c_{22} = \frac{\alpha\tilde{n}^D e^{\alpha\tilde{n}^D/(1+\tilde{z}^D)}}{(1+\tilde{z}^D)} + \tilde{r}^D - \tilde{\phi}^D,\ c_{23} = 1,\ c_{24} = \frac{\tilde{q}_K^D - \tilde{n}^D}{h_1}.$$

$$c_{31} = -\frac{\alpha\tilde{c}^D\tilde{n}^D e^{\alpha\tilde{n}^D/(1+\tilde{z}^D)}}{(1-\gamma)(1+\tilde{z}^D)^2},\ c_{32} = \frac{(1+\tau_y^D)\alpha e^{\alpha\tilde{n}^D/(1+\tilde{z}^D)}\tilde{c}^D}{(1-\gamma)(1+\tilde{z}^D)},\ c_{33} = 0,\ c_{34} = -\tilde{c}^D / h_1.$$

$$c_{41} = -\frac{\alpha \tilde{q}_K^D \tilde{n}^D e^{\alpha \tilde{n}^D/(1+\tilde{z}^D)}}{(1+\tilde{z}^D)^2} - bA^{-\rho}(1+\rho)(1-\tau_y^D)(\tilde{y}^D)^{\rho}\left(\frac{\partial \tilde{y}^D}{\partial \tilde{z}^D}\right),\ c_{42} = \frac{\alpha(1+\tau_y^D)\tilde{q}_K^D e^{\alpha \tilde{n}^D/(1+\tilde{z}^D)}}{(1+\tilde{z}^D)},\ c_{43} = 0,$$

$$c_{44} = (1+\tau_y^D)\tilde{r}^D - \tilde{\phi}^D,$$

where, $\frac{\partial \tilde{y}^D}{\partial \tilde{z}^D} = (1-b)A^{-\rho}\left(\frac{\tilde{y}^D}{\tilde{z}^D}\right)^{1+\rho}$.

Appendix B

The Provision of Public Capital and Optimal Fiscal Policy

An important objective for a government in a decentralized economy might be to maximize economic welfare by ensuring the provision of an optimal amount of public investment, either directly or through private providers. This requires the design of an optimal tax and subsidy policy that will replicate the (first-best) social optimum in a centrally planned economy. To solve the problem of time inconsistency, we introduce lump sum taxes into the framework, and examine how the particular mode of providing public capital can influence the design of optimal fiscal policy.

B1. The First-Best Equilibrium

The following set of equations describes the evolution of a centrally planned economy, where the social planner's resource allocation decision internalizes both the production and the borrowing externalities (all dynamic variables for the centrally planned economy are denoted by a superscript "*"):

$$\frac{\dot{z}^*}{z^*} = \left[\frac{q_G^* - 1}{h_2} - \delta_G\right] - \left[\frac{q_K^* - 1}{h_1} - \delta_K\right] \quad \text{(B1.1a)}$$

$$\frac{\dot{n}^*}{n^*} = r\left(n^*, z^*\right) + \frac{1}{n^*}\left[c^* + \frac{\left(q_K^*\right)^2 - 1}{2h_1} + \frac{\left\{\left(q_G^*\right)^2 - 1\right\} z^*}{2h_2} - y^*\right] - \left[\frac{q_K^* - 1}{h_1} - \delta_K\right] \quad \text{(B1.1b)}$$

$$\frac{\dot{c}^*}{c^*} = \left[\frac{r\left(n^*, z^*\right) + \left\{n^*/\left(1+z^*\right)\right\} r'(.) - \beta}{(1-\gamma)}\right] - \left[\frac{q_K^* - 1}{h_1} - \delta_K\right] \quad \text{(B1.1c)}$$

$$\frac{\dot{q}_K^*}{q_K^*} + \frac{bA^{-\rho}\left(y^*\right)^{1+\rho}}{q_K^*} + \frac{\left(q_K^* - 1\right)^2}{2h_1 q_K^*} + \left(\frac{n^*}{1+z^*}\right)^2 \frac{r'(.)}{q_K^*} - \delta_K = r\left(n^*, z^*\right) + \left(\frac{n^*}{1+z^*}\right) r'(.) \quad \text{(B1.1d)}$$

$$\frac{\dot{q}_G^*}{q_G^*}+\frac{(1-b)\,A^{-\rho}\left(y^*/z^*\right)^{1+\rho}}{q_G^*}+\frac{\left\{q_G^*-1\right\}^2}{2h_2 q_G^*}+\left(\frac{n^*}{1+z^*}\right)^2\frac{r'(.)}{q_G^*}-\delta_G=r\left(n^*,z^*\right)+\left(\frac{n^*}{1+z^*}\right)r'(.) \qquad \text{(B1.1e)}$$

The central planner corrects for three sources of externalities. First, from (B1.1c) we see that, in performing his optimization, the planner takes into account the marginal cost of borrowing from international capital markets. This is also reflected in the second term on the right hand sides of (B1.1d) and (B1.1e). Second, the third term on the left hand sides of (B1.1d) and (B1.1e) reflect the fact that the accumulation of both types of capital enhances the economy's debt-servicing capacity by reducing its debt-capital ratio. This raises the return from each type of capital by reducing the cost of borrowing. Finally, the central planner internalizes the social benefits of public capital formation and the interaction of the shadow prices of the two types of capital in making his allocation decisions. Therefore, irrespective of the mode of provision of public capital, replicating the first-best solution in a decentralized economy requires a fiscal policy that can correct these three sources of distortions.

B2. Optimal Fiscal Policy under Private Provision

In an economy with private provision, the government can use three policy instruments to achieve the first-best optimum: a tax on borrowing, τ_N^L, a tax on output, τ_Y^L, and a tied subsidy to the private agent, s^L, for investment in public capital. Introducing a lump sum tax T^L enables us to re-write the budget constraint (7) in the following manner

$$s^L\Omega\left(G^L/K_G^L\right)=\tau_Y^L Y^L+\tau_N^L\; r\left(N^L/K_T^L\right)N^L+T^L \qquad \text{(B2.1)}$$

A first-best tax and subsidy policy is defined as the set of policy instruments $\left(\hat{\tau}_N^d,\ \hat{\tau}_Y^d,\ \hat{s}^d\right)$ that not only replicates the steady-state equilibrium in a centrally planned economy, but also its dynamic adjustment path. Comparing the steady-state versions of (13a)-(13e) with (B1.1a)-(B1.1e), we get the following set of equations (note that to denote optimal levels, we have dropped the superscripts "*L*" and "*" for the dynamic variables):

$$\frac{\left(1+\hat{\tau}_N^L\right)\, r\left(\tilde{n},\tilde{z}\right)-\beta}{(1-\gamma)}=\frac{r\left(\tilde{n},\tilde{z}\right)+\left\{\tilde{n}/(1+\tilde{z})\right\}r'(.)-\beta}{(1-\gamma)} \qquad \text{(B2.2a)}$$

$$\left(1-\hat{\tau}_Y^L\right)\, bA^{-\rho}\left(\tilde{y}\right)^{1+\rho}=bA^{-\rho}\left(\tilde{y}\right)^{1+\rho}+\left(\frac{\tilde{n}}{1+\tilde{z}}\right)^2\frac{r'(.)}{\tilde{q}_K} \qquad \text{(B2.2b)}$$

$$\left(1-\hat{\tau}_Y^L\right)\left(1-b-\eta\right)A^{-\rho}\left(\frac{\tilde{y}}{\tilde{z}}\right)^{1+\rho}+\frac{\left\{\tilde{q}_G-\left(1-\tilde{s}\ \right)\right\}^2}{2\left(1-\tilde{s}\ \right)h_2}$$

$$=\left(1-b\right)A^{-\rho}\left(\frac{\tilde{y}}{\tilde{z}}\right)^{1+\rho}+\frac{\left\{\tilde{q}_G-1\right\}^2}{2h_2}+\left(\frac{\tilde{n}}{1+\tilde{z}}\right)^2 r'\left(.\right) \tag{B2.2c}$$

Equations (B2.2a)-(B2.2c) can be solved for the optimal values of the three policy instruments:

$$\hat{\tau}_N^L=\left(\frac{\tilde{n}}{1+\tilde{z}}\right)\frac{r'\left(\tilde{n},\tilde{z}\right)}{r\left(\tilde{n},\tilde{z}\right)} \tag{B2.3a}$$

$$\hat{\tau}_Y^L=-\left(\frac{\tilde{n}}{1+\tilde{z}}\right)^2\frac{r'\left(\tilde{n},\tilde{z}\right)}{bA^{-\rho}\tilde{y}^{1+\rho}} \tag{B2.3b}$$

$$\hat{s}^L=\frac{1}{2}\left[\left\{\left(\tilde{q}_G^2-1+\Sigma\right)^2-4\Sigma\right\}^{1/2}-\left(\tilde{q}_G^2-1+\Sigma\right)\right]$$

(B2.3c)

where, $\Sigma=2h_2\left[\left\{\eta-\left(1-b-\eta\right)\left(\frac{\tilde{n}}{1+\tilde{z}}\right)^2\frac{r'\left(\tilde{n},\tilde{z}\right)}{bA^{-\rho}\tilde{y}^{1+\rho}}\right\}A^{-\rho}\left(\frac{\tilde{y}}{\tilde{z}}\right)^{1+\rho}-\left(\frac{\tilde{n}}{1+\tilde{z}}\right)^2 r'\left(\tilde{n},\tilde{z}\right)\right]$.

The optimal fiscal policy package described in (B2.3a)-(B2.3c) corrects for both the external effects of public capital accumulation as well as capital market imperfections. The optimal tax on foreign borrowing, given in (B2.3a), corrects the borrowing externality by taxing foreign debt at a rate equal to the elasticity of foreign debt with respect to the debt-capital ratio. The optimal tax on income, given in (B2.3b), turns out to be a subsidy, reflecting a reward to the private agent for accumulating private capital, which improves the economy's aggregate debt-servicing capability. As a result, it is tied to the reduction in the marginal cost of borrowing due to investment in private capital, valued by its marginal product. Conversely, if the private agent were a net creditor to the rest of the world, then the appropriate policy would be to impose a tax on income. The optimal subsidy for public investment, given in (B2.3c), corrects for three distortions. First, it enables the private agent to internalize the social benefits from the aggregate stock of public capital. Second, it rewards the agent for accumulating public capital, which improves the economy's debt-servicing capability. Third, it takes into account the effect of the accumulation of private capital on the shadow price of public capital, q_G.

Finally, we should note that the optimal tax and subsidy policy is constant over time. By substituting (26a)-(26c) in (14), it can be easily verified that an economy with private provision of public capital can not only replicate the steady-state equilibrium in a centrally planned economy, but also its dynamic adjustment path.

B3. Optimal Fiscal Policy under Government Provision

In the presence of a lump sum tax, T^D, the flow budget constraint in an economy with direct government provision of public capital can be written as

$$\Omega\left(G^D / K_G^D\right) = \tau_Y^D Y^D + \tau_N^D \; r\left(N^D / K_T^D\right) N^D + T^D \tag{B3.1}$$

Then, a first-best taxation policy is defined as the set of policy instruments $\left(\hat{\tau}_N^D,\ \hat{\tau}_Y^D\right)$ that replicate the equilibrium for a centrally planned economy.

To characterize the first-best taxation policy we compare the steady-state versions of (16a) and (16d) with (B1.1a) and (B1.1d). This gives the following set of optimal tax rates:

$$\hat{\tau}_N^D = \left(\frac{\tilde{n}}{1+\tilde{z}}\right) \frac{r'\left(\tilde{n},\tilde{z}\right)}{r\left(\tilde{n},\tilde{z}\right)} \tag{B3.2a}$$

$$\hat{\tau}_Y^D = -\tilde{v}^D g - \left(\frac{\tilde{n}}{1+\tilde{z}}\right)^2 \frac{r'\left(\tilde{n},\tilde{z}\right)}{bA^{-\rho}\tilde{y}^{1+\rho}} \tag{B3.2b}$$

Setting the two tax rates in accordance with (B3.2a) and (B3.2b) will enable the decentralized economy to replicate the steady-state equilibrium of a centrally planned economy. The optimal tax on foreign borrowing, given by (B3.2a), is exactly similar to that in the economy with private provision. However, from (B3.2b) we see that the income tax rate is slightly different from the one in specified in (B2.2b). This is because direct government provision of public capital in a decentralized economy implies that the allocation of output to public investment, g, is arbitrary and therefore may be above, below, or equal to its social optimum.[2] Hence, the first component of the income tax rate specified in (28b) corrects for this deviation. For example, when $g < g^*$ (the social optimum), the shadow value of allocating an extra unit of output to public investment is positive, i.e., $\tilde{v}^D > 0$. As a result, income should be subsidized to encourage private capital accumulation, which by increasing the flow of output

[2] The expression in (B3.2b) is derived under the assumption that the central planner sets $g^* = g$ arbitrarily, to enable a comparison with direct government provision in a decentralized economy.

also increases the stock of public capital. The second component of the income tax is similar to (B3.2b) and reflects the reward to the private agent for accumulating private capital, which improves the economy's aggregate debt-servicing capability.

There is one important caveat to the first-best policy described in (B3.2a) and (B3.2b). If the government maintains $\tau_Y^D = \hat{\tau}_Y^D$ during the transition to the steady-state equilibrium, the adjustment path followed by the decentralized economy will fail to replicate that of its centrally planned counterpart. Therefore, $\hat{\tau}_Y^D$ is the first-best tax rate only in the steady state, but not in transition. To see this, let us denote the linearized matrix for the central planner's equilibrium dynamics in (B1.1a)-(B1.1e) by Δ^*. Then the eigenvalues in the centrally planned economy are the unique solutions to the polynomial

$$F\left(\mu^*\right) \equiv \text{Det}\left[\Delta^* - \mu^* I\right] = 0 \tag{B3.3}$$

where I is a 5x5 identity matrix. The equilibrium dynamics will be characterized by two stable (negative) eigenvalues and, for the centrally planned economy, we will denote these by $\mu_1^* < 0$ and $\mu_2^* < 0$. Then, from (B3.3) it must be the case that

$$F\left(\mu_1^*\right) = F\left(\mu_2^*\right) = 0 \tag{B3.4}$$

Substituting for $\hat{\tau}_N^D$ and $\hat{\tau}_Y^D$ in the linearized matrix Δ^D for the decentralized economy, we get a result corresponding to (B3.3) and (B3.4):

$$J\left(\mu_1^D\right) = J\left(\mu_2^D\right) = 0 \tag{B3.5}$$

where $\mu_i^D \left(i = 1,2\right)$ are the stable eigenvalues and $J(.)$ is a polynomial corresponding to $F(.)$ in the decentralized model. It can be verified that[3]

$$F\left(\mu_i^D\right) \neq 0,\ i = 1,2 \tag{B3.6}$$

It then follows from (B3.5) and (B3.6) that $\mu_1^D \neq \mu_1^*$, and $\mu_2^D \neq \mu_2^*$. Therefore, if $\hat{\tau}_N^D$ and $\hat{\tau}_Y^D$ are fixed over time, the decentralized economy will converge to the first-best steady-state equilibrium at a rate that is non-optimal, relative to that of the centrally planned economy. This happens because when the

[3] Due to the analytical complexity of our model, (B3.6) was verified numerically. For a more complete proof in a related but simpler model, see Turnovsky (1997).

government directly provides public capital, the private agent takes it as exogenously given and therefore does not internalize the effect of its private investment decisions on the shadow price of public capital during transition. This externality is not accounted for by a time-invariant income tax rate.

To correct the above problem, let us modify the tax on private income to

$$\tau_Y^D(t) = \hat{\tau}_Y^D + \theta_1\left[z^D(t) - \tilde{z}\right] + \theta_2\left[n^D(t) - \tilde{n}\right] \tag{B3.7}$$

where θ_1 and θ_2 are constants to be determined. The income tax rate in (B3.7) is time-varying, and tracks the dynamic evolution of the economy as the stocks of public and private capital and debt change over time. As Turnovsky (1997) points out, a time-varying tax rate enables the private agent to track the dynamic adjustment of the shadow price of public capital. Moreover, θ_1 and θ_2 are only relevant along the transition path, and do not affect the steady-state equilibrium.

However, the appropriate determination of the constants θ_1 and θ_2 is crucial for the first-best tax policy to replicate the dynamic adjustment of a centrally planned economy. To ensure this, the government must set θ_1 and θ_2 such that

$$J\left(\mu_1^D, \theta_1\right) = F\left(\mu_1^D\right) = 0 \tag{B3.8a}$$

$$J\left(\mu_2^D, \theta_2\right) = F\left(\mu_2^D\right) = 0 \tag{B3.8b}$$

When θ_1 and θ_2 are chosen in accordance with (B3.8a) and (B3.8b), the speed of adjustment in the decentralized economy will replicate that of the centrally planned economy. Moreover, θ_1 and θ_2 are only relevant along the transition path, and do not affect the steady-state equilibrium. As $z^D(t) \to \tilde{z}$ and $n^D(t) \to \tilde{n}$, $\tau_Y^D(t)$ will converge to its long-run optimal rate, $\hat{\tau}_Y^D$.