



Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

How misleading is linearization? Evaluating the dynamics of the neoclassical growth model[☆]

Manoj Atolia^a, Santanu Chatterjee^b, Stephen J. Turnovsky^{c,*}

^a Florida State University, Tallahassee, FL 32306, USA

^b University of Georgia, Athens, GA 30602, USA

^c Department of Economics, University of Washington, Box 353330, Seattle, WA 98195-3330, USA

ARTICLE INFO

Available online 23 June 2010

JEL classification:

E62

O41

C63

Keywords:

Public expenditure

Growth

Nonlinearities

Welfare analysis

ABSTRACT

This paper investigates the reliability of employing linearization to evaluate the dynamic adjustments to changes in productive government spending in a Ramsey growth model. If government expenditure is introduced as a flow and the dynamic adjustment is fast, linearization may yield a reasonably good approximation to the true dynamics, even for fairly large policy shocks. If government expenditure assumes the form of a stock, leading to more sluggish adjustment, linearization may yield misleading predictions. These errors occur at the beginning of the transition and weigh heavily in welfare calculations. The implications for temporary shocks and the speed of convergence are also considered.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

Since the early 1980s, the representative agent model has become the standard workhorse macro-model.¹ Being based on intertemporal optimization, the macroeconomic equilibrium it yields is inevitably *nonlinear* and saddle path stable. The nonlinearity arises from the diminishing marginal utility on the consumption side and the diminishing marginal productivity on the output side, whereas saddle-path stability is the outcome of the forward-looking nature of the intertemporal optimization.² While this combination of nonlinearity and saddle-path stability does not create much of a problem for characterizing the static short-run or steady-state equilibrium, for which well developed analytical and numerical algorithms exist, it does pose severe difficulties for the analysis of the transitional dynamics of the system.

The standard procedure for analyzing the transitional dynamics following a structural change or policy shock, therefore, is to linearize the dynamic system around its (post-shock) steady state and then to study the dynamic properties of this

[☆] This is a revised version of a paper presented at the 14th Annual Conference on Computing in Economics and Finance organized by the Society for Computational Economics, in Paris, June 26–28, 2008. We are grateful to Berc Rustem for making us aware of his early work in this area and for his encouragement. Turnovsky's research was supported in part by the Castor endowment at the University of Washington. The constructive suggestions of two referees are also gratefully acknowledged.

* Corresponding author. Tel.: +1 206 685 8028; fax: +1 206 685 7477.

E-mail address: sturn@u.washington.edu (S.J. Turnovsky).

¹ See e.g. Blanchard and Fischer (1989) and Turnovsky (2000) for textbook treatments.

² The saddle-path stability manifests itself in the requirement about the number of the positive and negative eigenvalues of the associated linear system. See Sargent (1979), Blanchard and Fischer (1989), Azariadis (1993), Agenor and Montiel (1996), and Turnovsky (2000) for more detailed textbook expositions spanning a wide variety of applications.

linearized system as an *approximation* to those of the true nonlinear system. This raises the central question of this paper: How reliable (or unreliable) is the linear approximation, especially when applied to nonlinear growth models for the purposes of evaluating the dynamic response of an economy to policy changes or structural shocks? While we know that linearization involves errors, how serious are they? These questions assume critical significance when one considers that one of the basic objectives of micro-based macro-dynamic models is to analyze the consequences of policy for intertemporal welfare.

Clearly, as long as the structural or policy changes are small, the linear approximation should be adequate and give reasonably accurate guidance to the “true” dynamic response of the economy. But one can easily envision scenarios where this need not be so. For example, if the economy undergoes large structural adjustments or if its dynamic evolution is characterized by an overall sluggish adjustment process, the economy may remain far from its long-run equilibrium for extended periods of time. In either case, to analyze the dynamic response of the economy by linearizing around its steady-state equilibrium may potentially be quite misleading. Yet this is the dominant practice adopted in the macro-growth literature, and has been widely applied in a range of areas, where the concern has been to characterize the specific transitional path to the steady state.

While conventional restrictions on the utility and production functions suffice to provide a solid understanding of the global qualitative properties of the transitional behavior of the standard one-sector Ramsey model, explicit computation of the transitional path becomes necessary if one wishes to quantify the impacts of specific policy or structural changes on the performance of the economy. Furthermore, numerical simulations of the transitional dynamics also become inevitable as the specification of the macro-dynamic system is enriched, and its dimensionality increases.³ Although saddle-point equilibria associated with intertemporal optimizing models are difficult to compute numerically (see Judd, 1998), explaining the popularity of linearization as the dominant technique for dynamic analysis, recently Atolia and Buffie (2009b) have sought to reduce the computational difficulty for dynamic models containing up to four state variables.⁴

An interesting (and, with increased access to user-friendly computational techniques, also practical) question that arises from all this is: how misleading is the process of linearization of the dynamic system for characterizing the transition path? Does it yield implications that are not only quantitatively, but also qualitatively, erroneous? The purpose of this paper, therefore, is to compare the numerical predictions of a linearized dynamic model with those implied by a nonlinear solution technique. In this respect it is related to earlier work by Karakitsos and Rustem (1989), who characterized the unreliability of linear approximations relative to nonlinear algorithms for versions of the IS–LM model. More recently, Becker et al. (2007) compare the performance of dynamic programming and second-order approximation methods for dynamic general equilibrium models by employing a model for which an exact analytical solution is available. However, our approach differs from Karakitsos and Rustem (1989) in that we consider models based on intertemporal optimization. Moreover, by employing a more general model for which no closed-form analytical solution is available, it also contrasts with Becker et al. (2007). Instead, our comparison is with a very accurate numerical solution obtained by employing shooting techniques. We restrict attention to an evaluation of the performance of linearization, as it is the dominant method in literature on growth and development.⁵

Since the difference between linear and nonlinear solution techniques will depend critically on the intrinsic structure of a model, it is difficult to derive general conclusions about the implications of linearization. Thus, while our approach is illustrative, nevertheless we are able to identify some very robust patterns arising from linearization in the context of our model.

The framework we employ is a variant of the basic one-sector neoclassical Ramsey growth model, augmented to allow for productive government spending. We consider two widely adopted versions of the model. In the first, government spending is introduced as a *flow* in production, as is the case in much of the recent endogenous growth literature, spawned by Barro's (1990) seminal contribution. In this case, the macro-dynamic equilibrium is generated by a one-dimensional nonlinear differential equation in the private capital stock. However, this specification has been subject to the criticism that since productive government spending falls largely on the economy's infrastructure, it is more appropriately represented as a *stock* rather than as a flow. Hence, the second formulation of the model introduces public capital as a stock, along with private capital, so that the macroeconomic equilibrium now involves two state variables.⁶

In both cases, we first calibrate the model and then solve it numerically using both linear approximation and nonlinear techniques. We introduce an increase in the rate of productive government spending. In order to test the sensitivity to linearization, we consider two alternative increases in the rate of public investment: (i) from 4% to 5%, (as a share of output) and (ii) from 4% to 8%. The first case, which we consider to be a “moderate” increase, represents a 25% increase in

³ The use of numerical methods is of course central to the real business cycle methodology; see e.g. Cooley (1995).

⁴ The numerical solution techniques necessary to solve differential equation systems have improved significantly in recent years. Most of these have involved some form of “shooting algorithm” and projection methods; see Judd (1998) for an extensive discussion. Forward-shooting algorithms, which involve a guess for the initial value of the jump variable, may lead to substantial errors along the transition path, especially where multiple jump variables are present. Reverse-shooting algorithms, recommended by Judd (1998) and Brunner and Strulik (2002), while attractive, have been difficult to apply due to the limited progress in developing algorithms for systems containing multiple state variables.

⁵ As noted in the final section, as part of the robustness check on our procedure we did run some simulations employing log-linearization. We also note that Kim and Kim (2003) compare first-order approximations to second-order approximations.

⁶ See e.g. Arrow and Kurz (1970), Baxter and King (1993), Fisher and Turnovsky (1998), and Turnovsky (2004).

the policy variable (the current level of government expenditure), while the second is a doubling of the rate of public investment and we characterize this as a “large” increase. We then compare the dynamic time paths of the key variables of the model and the welfare changes implied by the linear and nonlinear approximation methods for solving for the transitional paths. Since the nonlinear dynamic paths are the closest approximation to the “true” or global dynamic path, any deviation of the “linearized” dynamic paths from the nonlinear paths is characterized as an “error”.

As expected, we find that the farther an economy is from its steady-state equilibrium, the larger are the errors generated by linearization, both qualitatively and quantitatively. The time and distance from steady state (and therefore the associated errors from linearization) are closely related to the speed of convergence, and in this regard we find that the two policy specifications yield significant differences. When government spending is introduced as a flow, the transition is shorter, and the economy converges more rapidly to its steady-state equilibrium (due to the one-dimensional transition path). The errors from linearization are relatively small and decline quite rapidly as the economy approaches its new steady state.

In contrast, when government expenditure is introduced in the form of public capital (stock), the errors from linearization are more substantial. The linearized model consistently *over-predicts* consumption and welfare gains from an increase in public investment in infrastructure. Errors are much larger and are concentrated in the initial phase of the dynamic adjustment. Because the introduction of public capital as a second state variable significantly reduces the speed of convergence, this can compound to substantial long-run errors in computing welfare. Furthermore, in many cases, we find that the errors may even be *qualitative*: for example, when the “true” (nonlinear) solution implies a short-run decline in consumption, the linearized approximation predicts an increase. If this occurs, the use of linearization methods may especially yield quite misleading predictions for quantities such as intertemporal welfare, which place heavy weight on the instantaneous responses of consumption and labor supply.

We also compare the predictions of the linearized and nonlinear solutions when government spending shocks are temporary in nature, thus focusing on the effect of the *duration* of a shock for the underlying solution technique. We find that although the errors from linearizing the flow model remain tolerable for the Cobb–Douglas case, the magnitudes of the errors increase for higher values of the elasticity of substitution in production. For the stock model, the errors from linearization are more magnified compared to those generated by permanent shocks.

Finally, we also examine the consequences of linearization for the speed of convergence, a subject of extensive theoretical and empirical investigation in recent years.⁷ Empirical estimates of the speed of convergence have typically been much lower than those implied by theory, where a frequently used measure is the stable eigenvalue of a linearized dynamic system. Interestingly, we find that linearized models tend to *over-predict* the speed of convergence for substantial portions of the transition path. Moreover, they also provide qualitatively contrasting time profiles for the speed of adjustment relative to the underlying nonlinear models. Indeed, in a recent paper, Papageorgiou and Perez-Sebastian (2007) point out that the asymptotic speed of convergence implied by a linearized two-sector growth model is not able to reproduce actual convergence experiences of countries such as Japan and South Korea. Their results lend further support to our finding that the speed of convergence implied by linearization can be a misleading indicator of the *true* speed of convergence of a dynamic system.

The remainder of the paper is as follows. In Sections 2 and 3 we lay out the theoretical models underlying our analysis. Section 4 discusses some general issues related to numerical approximation. Section 5 presents the numerical comparisons between linearized and nonlinear solutions to the theoretical models in response to a permanent increase in government expenditure, while Section 6 presents similar comparisons for temporary shocks. Section 7 briefly addresses the issue of convergence, while Section 8 concludes with a brief overview of our findings.

2. The one state variable case: public investment as a “flow”

We begin with the simpler and more common specification, where productive government expenditure is introduced as a flow in the production function, as in Barro (1990) and much of the subsequent endogenous growth literature.

2.1. Analytical framework

Consider an infinitely lived representative agent, modeled as a consumer–producer, who maximizes utility from consumption (C) and leisure (l):

$$\text{Maximize } \int_0^{\infty} \frac{1}{\gamma} (Cl^\theta)^\gamma e^{-\beta t} dt, \quad -\infty < \gamma < 1, \quad \theta > 0 \quad (1)$$

The agent’s optimization is subject to the following budget constraint:

$$\dot{K} = (1-\tau)Y - C - T - \delta_K K \quad (2)$$

⁷ See Mankiw et al. (1992), Barro and Sala-i-Martin (1992) for early empirical work on this issue. Theoretical studies by Ortiguera and Santos (1997), Eicher and Turnovsky (1999), and Chatterjee (2005) introduce various modifications (such as installation costs, multiple capital goods, and capital utilization decisions) into standard growth models that reduce the speed of convergence to those implied by empirical studies.

where K denotes the private capital stock that depreciates at the constant rate δ_K , τ the rate of income tax, T lump-sum taxation, and Y the flow of output produced by the agent using a Constant Elasticity of Substitution (CES) production technology⁸:

$$Y \equiv Y(K, l) = A[\alpha\{C^\eta(1-l)\}^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho}, \quad A > 0, \quad 0 < \alpha < 1, \quad 0 < \eta < 1, \quad -1 < \rho < \infty \quad (3)$$

The production function is homogeneous of degree one with respect to private capital, K , and labor supply, $(1-l)$. Productive government expenditure, G , is introduced as a flow that interacts directly with labor to enhance its productivity.

The flow of public expenditures is specified as a fixed fraction of output

$$G = gY, \quad 0 < g < 1 \quad (4a)$$

with an increase in g specifying an expansion in expenditure. This specification is natural in a growth context. The government finances its expenditure using a combination of a proportional income tax and lump-sum taxation, while maintaining a balanced budget at all points of time:

$$G = \tau Y + T \quad (4b)$$

Combining (2), (4a), and (4b), yields the goods market equilibrium condition

$$\dot{K} = (1-g)Y - C - \delta_K K \quad (4c)$$

2.2. Consumer optimization

Performing the optimization yields the following standard first-order conditions:

$$C^{\gamma-1} l^{\theta\gamma} = \lambda \quad (5a)$$

$$\theta C^\gamma l^{\theta\gamma-1} = -\lambda(1-\tau) \frac{\partial Y}{\partial l} = \lambda(1-\tau)\alpha(Ag^\eta)^{-\rho} \frac{Y^{1+\rho(1-\eta)}}{(1-l)^{1+\rho}} \quad (5b)$$

$$\beta - \frac{\dot{\lambda}}{\lambda} = (1-\tau) \frac{\partial Y}{\partial K} - \delta_K = (1-\tau)(1-\alpha)A^{-\rho} \left(\frac{Y}{K}\right)^{1+\rho} - \delta_K \quad (5c)$$

together with the transversality condition

$$\lim_{t \rightarrow \infty} \lambda K e^{-\beta t} = 0 \quad (5d)$$

Eq. (5a) equates the marginal utility of consumption to the shadow value of capital, (5b) equates the marginal utility of leisure to the utility of foregone income, while (5c) equates the marginal utility of consumption to the after-tax rate of return on capital.

2.3. Equilibrium dynamics

The equilibrium dynamics can be described in terms of the evolution of the stock of private capital, K , and leisure, l .⁹ It is straightforward to derive the dynamics of capital accumulation. We first divide (5b) by (5a) to get the marginal rate of substitution between consumption and leisure:

$$\frac{C}{l} = \frac{\alpha(Ag^\eta)^{-\rho}(1-\tau)}{\theta} \frac{Y^{1+\rho(1-\eta)}}{(1-l)^{1+\rho}} \quad (6)$$

Combining this expression with the production function enables us to express C in the form

$$C \equiv C(l, K; g, \tau)$$

which on substituting into (4c) yields

$$\dot{K} = (1-g)Y(K, l) - C(l, K; g, \tau) - \delta_K K \quad (7a)$$

The derivation of the dynamics of l is more involved. First, we differentiate (5a) with respect to time, and combine with (5c), to obtain:

$$(\gamma-1) \frac{\dot{C}}{C} + \theta\gamma \frac{\dot{l}}{l} = \beta + \delta_K - (1-\tau)(1-\alpha)A^{-\rho} \left(\frac{Y}{K}\right)^{1+\rho} \quad (8)$$

⁸ This formulation serves as an adequate vehicle to illustrate the computational issues we wish to discuss. The introduction of government spending as labor-augmenting has the convenience of ensuring that the economy has a balanced growth path if the model is extended to introduce population growth or technological change. For the Cobb–Douglas production function, which serves as our benchmark, how government spending is introduced is unimportant.

⁹ The structure is virtually identical to that of Turnovsky (2002), except that the economy has been augmented by the addition of a government fiscal sector.

Next, differentiating (6) with respect to time gives

$$\frac{\dot{C}}{C} - \frac{\dot{l}}{l} = (1 + \rho) \left(\frac{l}{1-l} \right) \frac{\dot{l}}{l} + [1 + \rho(1-\eta)] \frac{\dot{Y}}{Y} \tag{9}$$

and taking the time derivative of the production function yields

$$\frac{\dot{Y}}{Y} = A^{-\rho} \left[\frac{g^{\eta\rho}(1-l)^\rho}{g^{\eta\rho}(1-l)^\rho - A^{-\rho}\alpha\eta Y^{\rho(1-\eta)}} \right] \left[\frac{\alpha Y^{\rho(1-\eta)} l}{g^{\eta\rho}(1-l)^\rho} \frac{\dot{l}}{l} + (1-\alpha) \left(\frac{Y}{K} \right)^\rho \frac{\dot{K}}{K} \right] \tag{10}$$

Eliminating \dot{C}/C and \dot{Y}/Y from (8) to (10), we can express the dynamics for leisure as

$$\dot{l} = \frac{J(l, K, \dot{K})}{H(l, K)} \tag{11}$$

where

$$J(l, K, \dot{K}) = l \left[\beta + \delta_K - (1-\tau)(1-\alpha)A^{-\rho} \left(\frac{Y}{K} \right)^{1+\rho} + (1-\gamma)(1-\alpha)\{1 + \rho(1-\eta)\}A^{-\rho} \left(\frac{Y}{K} \right)^\rho \left(\frac{\dot{K}}{K} \right) \right] \tag{12a}$$

$$H(l, K) = [\gamma(1+\theta)-1] - (1-\gamma) \left[(1+\rho) \left(\frac{l}{1-l} \right) + \{1 + \rho(1-\eta)\} \left\{ \frac{A^{-\rho}\alpha Y^{\rho(1-\eta)}}{g^{\eta\rho}(1-l)^\rho - \alpha\eta A^{-\rho} Y^{\rho(1-\eta)}} \right\} \right] \tag{12b}$$

Finally, using (7a) to substitute for \dot{K} in (11) yields an equation for i of the form

$$\dot{l} = \frac{A(l, K)}{B(l, K)} \tag{7b}$$

Eqs. (7a) and (7b) are a highly nonlinear pair of autonomous differential equations in K and l . This is the conventional Ramsey model, modified by endogenous labor supply, and augmented with productive government spending. The standard procedure is to linearize these equations about the steady-state given in (13) below, when it can be shown to be a saddle-point, with a one-dimensional stable manifold. Our task is to assess the inaccuracies associated with this approximation.

2.4. Steady-state equilibrium

The steady-state equilibrium is attained when $\dot{K} = \dot{l} = 0$. The corresponding conditions are

$$(1-g)\tilde{Y} = C(\tilde{l}, \tilde{K}) + \delta_K \tilde{K} \tag{13a}$$

$$(1-\tau)(1-\alpha)A^{-\rho} \left(\tilde{Y}/\tilde{K} \right)^{1+\rho} = \beta + \delta_K \tag{13b}$$

$$\tilde{Y} \equiv Y(\tilde{l}, \tilde{K}) = A[\alpha\{(g\tilde{Y})^\eta(1-\tilde{l})\}^{-\rho} + (1-\alpha)\tilde{K}^{-\rho}]^{-1/\rho} \tag{13c}$$

which jointly determine the steady-state values \tilde{Y} , \tilde{K} and \tilde{l} . Once these are known, the steady-state levels of consumption can immediately be determined from (6).

3. The two state variable case: public investment as a “stock”

While introducing productive government expenditure as a flow is analytically convenient in the sense of limiting the dimension of the model, it may be an inadequate representation of public infrastructure, which is more appropriately represented as a capital stock. To incorporate this we modify the production function (3) to

$$Y = A[\alpha\{K_G^\eta(1-l)\}^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho}, \quad A > 0, \quad 0 < \alpha < 1, \quad 0 < \eta < 1 \tag{14}$$

where K_G represents the aggregate stock of public capital. The agent's optimization problem remains the maximization of (1) subject to (2), where the production function is now replaced by (14). The accumulation of public capital takes place according to

$$\dot{K}_G = G - \delta_G K_G = gY - \delta_G K_G, \quad 0 < \delta_G < 1 \tag{15}$$

where δ_G is the rate of depreciation.

The main change introduced by this re-specification of government expenditure is that the macro-dynamic equilibrium is represented by a third-order dynamic system involving the evolution of private capital, K , public capital, K_G , and leisure l . The corresponding dynamic equations are

$$\dot{K} = (1-g)Y - C(l, K, K_G) - \delta_K K \tag{16a}$$

$$\dot{K}_G = gY - \delta_G K_G \tag{16b}$$

$$\dot{l} = \frac{J(l, K, K_G, \dot{K}, \dot{K}_G)}{M(l, K, K_G)} \tag{16c}$$

where,

$$C \equiv C(l, K, K_G) = \frac{\alpha A^{-\rho} (1-\tau) l K_G^{-\eta \rho}}{\theta} \left[\frac{Y}{1-l} \right]^{1+\rho}$$

$$J(l, K, K_G) = l \left[\beta + \delta_K - (1-\tau)(1-\alpha) A^{-\rho} \left(\frac{Y}{K} \right)^{1+\rho} + (1-\gamma)(1+\rho) \left\{ A_1 \frac{\dot{K}}{K} + A_2 \frac{\dot{K}_G}{K_G} \right\} \right]$$

$$M(l, K, K_G) = \theta \gamma - (1-\gamma) \left[1 + (1+\rho) \left\{ \frac{l}{1-l} + \alpha A^{-\rho} \left(\frac{Y}{K_G^\eta (1-l)} \right)^\rho \right\} \right]$$

$$A_1 = (1-\alpha) A^{-\rho} \left(\frac{Y}{K} \right)^\rho, \quad A_2 = \eta \left[\alpha A^{-\rho} \left\{ \frac{Y}{K_G^\eta (1-l)} \right\}^\rho - \frac{\rho}{1+\rho} \right]$$

This system is a slight generalization of that analyzed by Turnovsky (2004).¹⁰ Formal analysis of the linearized approximation proves to be intractable, although Turnovsky found for extensive simulations that it yielded two negative and one positive eigenvalue, implying a unique stable adjustment path. In this case, the stable manifold, being two-dimensional, allows for much more flexible dynamic behavior and can be reconciled more broadly with empirical evidence.

The steady-state equilibrium, attained when $\dot{K} = \dot{K}_G = \dot{l} = 0$, is described by

$$(1-g)\tilde{Y} = C(\tilde{l}, \tilde{K}, \tilde{K}_G) + \delta_K \tilde{K} \tag{17a}$$

$$g\tilde{Y} = \delta_G K_G \tag{17b}$$

$$(1-\tau)(1-\alpha) A^{-\rho} (\tilde{Y}/\tilde{K})^{1+\rho} = \beta + \delta_K \tag{17c}$$

where

$$\tilde{Y} = A [\alpha (\tilde{K}_G^\eta (1-\tilde{l}))^{-\rho} + (1-\alpha) \tilde{K}^{-\rho}]^{-1/\rho}$$

$$\tilde{Y}/\tilde{K} = A \left[(1-\alpha) + \alpha \left\{ \tilde{K}_G^\eta (1-\tilde{l}) \tilde{K} \right\}^{-\rho} \right]^{-1/\rho}$$

Eqs. (17a)–(17c) can in principle be solved for \tilde{K} , \tilde{K}_G , and \tilde{l} .

4. Numerical analysis

As we have noted, the existing literature on growth and development, almost without exception, analyzes transitional dynamics using linearization, a procedure likely to be acceptable if the economy is subject to only small stationary shocks that keep it “close” to the steady state. While that may be true for the analysis of business cycles, applications in growth and development are more likely to consider issues such as policy reforms, leading to substantial structural changes; see Rodrik (1996). In this case linearization may cease to be reliable, and the need for a technique to evaluate the global nonlinear transition paths arises.

As mentioned in the introduction, there are several methods that can be employed to study nonlinear dynamics, though they have received surprisingly little or no attention in the field of growth and development. Atolia and Buffie (2009b) attempt to bridge the gap between existing numerical methods and their applications in macroeconomics. They have developed reliable and user-friendly programs that can be used by an average practitioner with experience in higher-level programming environments such as *Mathematica*, *Matlab*, etc.¹¹ For the experiments in Section 5 we employ one of the Atolia–Buffie (2009b) reverse-shooting programs (*Reverse Shoot 2D*). For the temporary shocks, we apply the forward-shooting method outlined in Atolia and Buffie (2009a).¹²

¹⁰ Turnovsky (2004) restricted the production function to be of the Cobb–Douglas form.

¹¹ The programs are based on the underlying principle of “shooting” and can be used to solve a wide-variety of models in growth and development containing 2–4 state variables, 2–3 jump variables, and a variety of shocks categorized by whether they are temporary or permanent. These programs are available in the public domain at <http://mailer.fsu.edu/~matolia/>. Also, with the availability of cheap computing power, computation times are quite reasonable. For lower-order systems, computing times are of the order of a few minutes to a few tens of minutes. For higher-order systems, it may stretch to several hours.

¹² We have also computed the dynamic paths for the nonlinear system using alternative projection methods and the results are the same as our shooting algorithm to the 5th decimal place. We are therefore confident that our nonlinear dynamic paths are very close to the true optimal path.

In evaluating the accuracy of linearization, a basic issue that we need to address concerns the assessment of the errors. In doing so, our objective is to present them in the most informative way. As our focus is on analyzing changes in the economy and evaluating the errors from linear approximation along a transitional path we prefer to measure the errors associated with linearization as a percentage of the initial pre-shock equilibrium value, namely by¹³

$$\frac{[x_i(T) - x(T)]}{x_0} \quad (18)$$

In any event, the graphical illustrations for the time paths give a good intuitive sense as to the accuracy of the linearization procedure and where it is most vulnerable.

At this point we should relate our approach to assessing errors to that based on Euler equation residuals, as applied to dynamic stochastic general equilibrium (DSGE) models; see e.g. Judd (1992), Den Haan and Marcet (1994), Santos (2000), and Aruoba et al. (2006). There are several differences both in the objective of our analysis and the formulation of the model that suggest that the Euler equation approach is both unnecessary and inappropriate in our context.

First, the main problem in assessing the accuracy of approximate numerical solutions to DSGE models is that usually the true or exact solution is unavailable. Thus, after having computed an approximate numerical solution, the best that can be done (to compare different methods or to evaluate the solution from a particular method) is to assess the size of the errors in the system equations (Euler equation, etc.), when this approximate solution is substituted back in. However, having a deterministic structure where we can in fact compute the exact solution, we do not have that constraint. Hence, we do not need to employ the indirect approach of finding the Euler equation errors and subjecting our analysis to the limitations that that procedure entails; see Santos (2000).

Second, the differences in our objectives vis-à-vis the DSGE literature suggests that the Euler equation approach is inappropriate for our purposes. Our focus is on the entire dynamic response of the economy to a policy shock, including the short and medium run. In contrast, the Euler equation is concerned with the asymptotic properties of the accuracy of the numerical solutions. That is, the approach is essentially able to provide only asymptotic bounds on errors in economic variables, given the errors in the Euler equation. Of course, this is relevant for DSGE models where interest centers on the correlation properties and other higher-order moments. But it is less so for a growth model where we are concerned with assessing impact of policy (and its accuracy) along the entire transitional path.

Moreover, errors in the Euler equation (which is a differential equation) do not fully capture the errors in the initial jumps of the control variables. In fact, when simulating a DSGE model after finding the policy function, the effect of initial conditions on the dynamic path is explicitly eliminated by discarding the initial simulated path. But in our context, we are explicitly concerned with the initial response of the economy to a policy shock since this is often the critical determinant of the entire subsequent dynamic adjustment. And, since the Euler equation only ties down the slope and not the initial condition for the jump variable, it is not hard to see that, small errors in the Euler equation at every step can accumulate into large errors in the impact response of the jump variable.¹⁴

5. Dynamics of government expenditure

5.1. The “flow” model: one state variable case

We begin by reporting the numerical results obtained for the flow version of the model set out in Section 2. These are based on the base parameters reported in Table 1, all of which are non-controversial. The rate of time preference, $\beta=0.04$, distributive share of labor, $\alpha=2/3$, the elasticity of leisure in utility, $\theta=1.75$, rate of depreciation, $\delta_K=0.05$, are all standard.¹⁵ The average tax rate, $\tau=0.20$, and rate of government investment expenditure of $g_0=0.04$, is consistent with empirical evidence, while the productivity elasticity of government spending $\eta=0.20$, is also conventional and plausible.¹⁶ Limited sensitivity analysis is conducted by varying the elasticity of substitution, $\sigma \equiv 1/(1+\rho)$, across 0.5, 1, 1.25, and the intertemporal elasticity of substitution in consumption between 1 and 0.40 [i.e. $\gamma=0, -1.5$]. These parameters yield the following steady states, reported in Table 2 (which are independent of γ). Given its prominent role in macro-dynamics and modern growth theory, the Cobb–Douglas function ($\sigma=1$) serves as a plausible benchmark, but our range is chosen in the light of recent empirical evidence based on different data sets and covers the range of plausible estimates.¹⁷

¹³ A seemingly natural measure of the error associated with a variable, x say, is the quantity $[x_i(T) - x(T)]/[x(T) - x_0]$. The problem is that this measure, which is expressed as a percentage of the true change, can misleadingly suggest huge errors, when in fact they are quite small. This is particularly true for the higher-order model which may generate non-monotonic behavior during the transition. Thus, if starting from x_0 , the true path for $x(t)$ crosses its initial value, x_0 during the transition, at some point T_1 , then around $t=T_1$, $[x_i(T) - x(T)]/[x(T) - x_0]$ will imply huge errors (infinite at $t=T_1$), even if $x_i(t)$ is tracking $x(t)$ closely.

¹⁴ We also note that, if the maximum Euler equation error is used as a measure of the errors, the use of Euler equation residuals in a continuous-time framework is likely to also over-estimate the errors significantly. This is because the “smoothing” over time which occurs with discrete time does not take place in this case.

¹⁵ See e.g. Cooley (1995) which summarizes the conventional parameters employed in the real business cycle literature.

¹⁶ In the United States total government spending is approximately 20% of GDP, of which about 20% (i.e. 4% of GDP) is devoted to public investment. Setting $\eta=0.20$ falls well within the range of evidence cited by Gramlich (1994).

¹⁷ See e.g. the early estimates of Berndt (1976) and the more recent evidence of Antràs (2004), Klump et al. (2007).

Table 1
Base parameters.

Taste parameters:	$\theta=1.75, \beta=0.04$
Technological parameters	$\alpha=0.667, A=1, \eta=0.20, \delta_K=0.05$
Policy parameters	$\tau=0.20, g_0=0.04$

Table 2
Initial steady states.

	\tilde{K}/\tilde{Y}	\tilde{C}/\tilde{Y}	\tilde{i}
$\sigma=0.5$	1.72	0.87	0.703
$\sigma=1$	2.96	0.81	0.727
$\sigma=1.25$	3.89	0.77	0.749

The exercise that is performed is to introduce two increases in g : (i) a “moderate” increase, from 0.04 to 0.05 which implies a 25% increase in G from its initial level, and (ii) a “large” increase, from 0.04 to 0.08, i.e. a 100% increase from the initial level of G .¹⁸ The resulting dynamic paths are computed first for the linearized system and then for the nonlinear system, using the reverse-shooting algorithm. As noted in Section 2, the system has only one state variable, K . Thus the linearized system, being saddle-point-stable, has only one negative eigenvalue and one independent jump variable.

The results for the errors in the time paths of key variables and in the corresponding welfare changes are summarized in Tables 3 and 4, respectively. Since both the initial and final steady states are calculated accurately, the errors due to linearization arise only along the transitional path. Table 3 summarizes the errors from linearization for a moderate increase in g (from 0.04 to 0.05), while Table 4 reports the corresponding errors for a large increase in g (from 0.04 to 0.08). As noted, the errors in the time paths of the linearized solutions (Tables 3 and 4, Panel A) are expressed as a *percentage of the initial steady-state value of the respective variable*. Panels B in Tables 3 and 4 report the welfare gains as the percentage increase in the permanent consumption flow [over the pre-shock level] that yields the same utility over time T as does the increase in g . The details of the welfare calculation are reported in the Appendix. Since changes in welfare have been transformed into equivalent percentage changes in consumption flows, the errors from linearization are more appropriately measured by (18) which expresses them as *percentage point* errors. But to illustrate the corresponding magnitudes of the errors expressed as actual percentages, we also report the latter (see footnote¹³).

As both Tables 3 and 4 yield the same pattern, we shall focus primarily on the former (a moderate increase in government spending). We first consider the case $\sigma=1$, and $\gamma=-1.5$, which serves as a commonly employed benchmark. This yields the eigenvalue $\lambda=-0.054$, implying an asymptotic speed of convergence of 5.4%, much higher than the early benchmark estimates of speeds of convergence of around 2–3% per annum; see e.g. Mankiw et al. (1992), Barro and Sala-i-Martin (1992).¹⁹ For the moderate fiscal expansion in Table 3, the errors are all less than 0.06% of their respective initial values, while when g is doubled (Table 4), they increase by about a factor of 11. In the case of income, the linear approximation overstates its initial increase by 0.056% of its pre-shock level, which is equivalent to a 1.25% error vis-a-vis the true increase, while, the overstatement of 0.655 in Table 4 is equivalent to a 4.23% error. Over time these decline steadily.

The initial increase in consumption, together with decline in leisure, leads to an initial welfare increase equivalent to 1.836% of the pre-shock consumption level. Because of the over-prediction of consumption, the linearized model predicts a 1.894% increase in welfare, which is a 0.058 percentage point error, equivalent to a 3.17% error. Accumulated over time these errors decline to 0.02 percentage points (3.657% vs. 3.637%), which is approximately a 0.57% error. If g is doubled from 0.04 to 0.08, these errors increase from 0.632 percentage points, on impact, to 0.233 percentage points, over time. Thus for the benchmark case, the errors committed by linearization do not seem to be misleading and should be acceptable. Even if the rate of expenditure is doubled, the estimated accumulated welfare effects from linearization overstate the true effect by only 1.23% (13.064 vs. 12.905).

Other interesting patterns that can be identified include the following:

- (1) More rapid convergence (e.g. when $\sigma=0.5$, and $\gamma=0$) is associated with larger initial errors, but these also decline more rapidly over time. However, when there is slower convergence (e.g. $\sigma=1.25$ and $\gamma=0$), the errors from linearization are much more persistent. The initial errors are most pronounced for consumption (the jump variable). Comparing

¹⁸ While these rates of public investment are arbitrary, the base rate of 4% is chosen as being close to the recent US experience. We have also introduced comparable increases in the rates of public investment, starting from an initial point nearer the optimum (which can be computed), to see whether the errors from linearization were sensitive to the deviation from the optimum. However, the base value of g had little impact on our findings.

¹⁹ Subsequent research, however, has questioned the accuracy of these original estimates, suggesting that they ignore a number of econometric issues, as a result of which they are downwardly biased. Once one controls for these factors, the estimates of the convergence rates both increase and become more sensitive to the time period, the set of countries and their stages of development; see e.g. Islam (1995), Caselli et al. (1996), Evans (1997), and Temple (1998).

Table 3
The “Flow” model (one state variable case).

Moderate increase in g : 0.04–0.05													
A. Errors at different points of time (T)													
	T	$\gamma=0$					$\gamma=-1.5$						
		0	2	5	10	20	50	0	2	5	10	20	50
$\sigma=0.50$		$\lambda_1 = -0.313$					$\lambda_1 = -0.140$						
	C	0.101	0.045	0.015	0.003	0.000	0.000	0.097	0.065	0.036	0.015	0.003	0.000
	l	0.038	0.005	-0.002	-0.001	0.000	0.000	0.035	0.020	0.009	0.002	0.000	0.000
	K	0.000	0.029	0.020	0.005	0.000	0.000	0.000	0.019	0.026	0.020	0.007	0.000
	Y	0.160	0.047	0.008	0.001	0.000	0.000	0.114	0.071	0.036	0.013	0.002	0.000
$\sigma=1.00$		$\lambda_1 = -0.135$					$\lambda_1 = -0.054$						
	C	0.048	0.032	0.017	0.007	0.001	0.000	0.053	0.045	0.036	0.025	0.012	0.002
	l	-0.044	-0.027	-0.013	-0.005	-0.001	0.000	-0.008	-0.007	-0.005	-0.004	-0.002	0.000
	K	0.000	0.007	0.009	0.007	0.002	0.000	0.000	0.006	0.012	0.016	0.015	0.004
	Y	0.057	0.034	0.016	0.005	0.001	0.000	0.056	0.048	0.037	0.025	0.012	0.002
$\sigma=1.25$		$\lambda_1 = -0.084$					$\lambda_1 = -0.033$						
	C	0.040	0.031	0.021	0.012	0.004	0.000	0.048	0.044	0.038	0.031	0.020	0.006
	l	-0.063	-0.046	-0.029	-0.014	-0.003	0.000	-0.019	-0.017	-0.014	-0.011	-0.006	-0.002
	K	0.000	0.004	0.008	0.008	0.005	0.000	0.000	0.004	0.008	0.013	0.016	0.010
	Y	0.044	0.032	0.021	0.011	0.003	0.000	0.047	0.043	0.038	0.030	0.020	0.006
B. Welfare gain on impact and in the long run (% of initial consumption)													
		$\gamma=0$					$\gamma=-1.5$						
		$T=0$		$T=\infty$			$T=0$		$T=\infty$				
$\sigma=0.50$	Linear			0.235	4.404		2.157		4.301				
	Nonlinear			0.164	4.395		2.087		4.287				
	%pt error			0.071	0.009		0.070		0.014				
	%Error			43.32	0.20		3.34		0.32				
$\sigma=1.00$	Linear			0.213	3.885		1.894		3.657				
	Nonlinear			0.136	3.872		1.836		3.637				
	%pt error			0.077	0.013		0.058		0.020				
	%Error			56.40	0.33		3.17		0.57				
$\sigma=1.25$	Linear			-0.050	3.382		1.494		3.071				
	Nonlinear			-0.126	3.364		1.436		3.043				
	%pt error			0.076	0.018		0.058		0.028				
	%Error			-60.52	0.54		4.07		0.92				

consumption for the logarithmic utility function, we see that the initial error is 0.101 for $\sigma=0.5$ ($\lambda = -0.313$) and only 0.048 if $\sigma=1$ ($\lambda = -0.135$). After 5 periods the errors in the former case will have dropped to 0.015 as compared to 0.017 in the latter case.

- (2) In contrast, the error in the sluggish variable, K , is zero initially (since jumps in the state variable are ruled out), and increases gradually for the first few periods along the transitional path, before declining.
- (3) The speed of convergence declines (and therefore the errors from linearization increase) as the elasticity of substitution in production increases. In contrast, the speed of convergence increases (and the errors from linearization decline) as the intertemporal elasticity of substitution increases.
- (4) Both consumption and leisure are over-estimated when σ is small. For larger σ , leisure tends to be under-estimated and consumption over-estimated, with the resulting errors in computing welfare being partially offsetting.
- (5) For moderate increases in g , the percentage point errors in measuring the welfare gains are relatively small, often of the order of 0.07 in the short run and reduced to 0.01 when taken over the entire infinite horizon. These errors increase by about a factor of 10 when g is doubled. Because of the small true changes in welfare in the short run, the percentage error committed by linearizing can be substantially larger. For example, for logarithmic utility and Cobb–Douglas production function linearization over-estimates the short-run welfare gain of 0.136 percentage points by 0.077, which is equivalent to an error of over 56% (of the true gain), though this falls dramatically over time

The overall message from these simulations for the flow specification of government expenditure is that the errors committed using linearization are generally tolerable, and in no cases lead to seriously misleading qualitative conclusions, particularly if one focuses on the longer run.

Table 4
The “Flow” model (one state variable case).

Large increase in g: 0.04–0.08													
A. Errors at different points of time (T)													
T	$\gamma=0$						$\gamma=-1.5$						
	0	2	5	10	20	50	0	2	5	10	20	50	
$\sigma=0.50$	$\lambda_1 = -0.039$						$\lambda_1 = -0.132$						
C	1.11	0.503	0.171	0.033	0.001	0.000	1.08	0.717	0.403	0.167	0.036	0.001	
I	0.372	0.048	-0.023	-0.009	0.000	0.000	0.362	0.208	0.091	0.023	0.001	0.000	
K	0.000	0.332	0.223	0.057	0.003	0.000	0.000	0.214	0.296	0.229	0.077	0.002	
Y	1.83	0.539	0.089	0.005	0.000	0.000	1.32	0.822	0.413	0.143	0.028	0.000	
$\sigma=1.00$	$\lambda_1 = -0.134$						$\lambda_1 = -0.054$						
C	0.533	0.348	0.192	0.078	0.016	0.000	0.592	0.503	0.397	0.272	0.135	0.022	
I	-0.460	-0.281	-0.139	-0.047	-0.008	-0.000	-0.088	-0.071	-0.055	-0.037	-0.017	-0.003	
K	0.000	0.081	0.111	0.084	0.027	0.001	0.000	0.070	0.137	0.183	0.169	0.048	
Y	0.669	0.397	0.187	0.058	0.008	0.000	0.655	0.549	0.425	0.282	0.132	0.019	
$\sigma=1.25$	$\lambda_1 = -0.084$						$\lambda_1 = -0.033$						
C	0.439	0.340	0.235	0.132	0.047	0.003	0.532	0.486	0.425	0.342	0.225	0.072	
I	-0.667	-0.481	-0.298	-0.139	-0.035	-0.001	-0.195	-0.173	-0.146	-0.110	-0.064	-0.015	
K	0.000	0.053	0.091	0.097	0.059	0.006	0.000	0.046	0.098	0.154	0.190	0.119	
Y	0.512	0.377	0.243	0.121	0.035	0.002	0.549	0.501	0.437	0.350	0.230	0.073	
B. Welfare gain on impact and in the long run (% of initial consumption)													
	$\gamma=0$						$\gamma=-1.5$						
	T=0			T=∞			T=0			T=∞			
$\sigma=0.50$													
Linear				-0.468			13.415			5.930			
Nonlinear				-1.234			13.313			5.165			
%pt error				0.766			0.102			0.765			
%Error				-62.10			0.77			14.80			
$\sigma=1.00$													
Linear				-0.497			11.656			5.118			
Nonlinear				-1.308			11.514			4.486			
%pt error				0.811			0.142			0.632			
%Error				-62.03			1.24			14.08			
$\sigma=1.25$													
Linear				-1.345			9.978			3.827			
Nonlinear				-2.157			9.776			3.192			
%pt error				0.812			0.202			0.635			
%Error				-37.65			2.07			19.89			

5.2. The “stock” model: two state variables case

We now turn to the stock version of the model outlined in Section 3. As in the flow case, we present results for a moderate and a large increase in g both for the linearized system and the actual nonlinear system. The parameterization of the model is same as that of the flow model. The only additional parameter is the rate of depreciation of public capital or infrastructure, δ_G , which is set to 0.035, following its calibration as in Turnovsky (2004).

For the sensitivity analysis, like the flow case, we allow the elasticity of substitution in production, σ , to vary across 0.5, 1, 1.25, and the intertemporal elasticity of substitution in consumption between 1 and 0.4 [i.e. $\gamma=0, -1.5$]. In all these cases, the initial steady state for the private variables is identical to that of the flow model (see Table 2). In addition, the ratio of public to private capital is 0.66, 0.39, 0.29, corresponding to the three elasticity measures $\sigma=0.5, 1, 1.25$. Further sensitivity analysis is also conducted for different degrees of external effects of infrastructure on production by allowing η to vary across 0.05 and 0.40 for the benchmark case with $\sigma=1, \gamma=-1.5$.²⁰ It may be noted that the system now has two state variables: the stocks of private and public capital. Accordingly, the dynamic system has two negative eigenvalues as required for saddle-path stability. For the benchmark case they are $\lambda_1 = -0.069$ and -0.025 .

²⁰ This does not affect the steady-state values listed in Table 2 for the benchmark case.

5.2.1. Moderate increase in government spending

We begin by presenting the results for the moderate increase in public investment in infrastructure, which corresponds to an increase in g from its initial value of 0.04–0.05. Tables 5A and B present the errors from linearization in predicting key variables, and the welfare gains, respectively, and are directly analogous to Tables 3A and B. In addition, Table 5C presents some sensitivity results for welfare as the importance of public capital in production is varied.

The time paths for the key variables for the benchmark case ($\sigma=1, \gamma=-1.5, \eta=0.2$) are illustrated in Fig. 1, where the solid line illustrates the “correct” nonlinear solution and the dashed line the linearized approximation.²¹ Focusing first on the solid line, we see that an increase in the rate of government investment immediately shifts resources away from the private sector, causing an initial decline in the rate of private investment. The increase in public investment immediately raises the productivity of labor causing an initial increase in labor supply, leading to an initial increase in output. However this is insufficient to meet the economy-wide demand and consumption immediately falls. Over time, as public capital is accumulated, productivity of private investment increases, leading to an overall expansion of the economy, including employment and consumption. This is the gist of Fig. 1(1–6).

It is also evident from Fig. 1 that while the linearized approximation yields a generally similar pattern, it over-predicts the short-run responses of all key variables. To see why this occurs it suffices to focus on the case of inelastic supply, when the following second-order approximations to the returns on capital and labor obtain

$$r \equiv F_K \simeq F_K^* + F_{KK}^*(K - K^*) + \frac{1}{2} F_{KKK}^*(K - K^*)^2 \tag{19a}$$

$$w \equiv F_L \simeq F_L^* + F_{LK}^*(K - K^*) + \frac{1}{2} F_{LKK}^*(K - K^*)^2 \tag{19b}$$

where for the Cobb–Douglas, as well as for plausible values of the elasticity of substitution, $F_{KKK} > 0, F_{LKK} < 0$. By ignoring the second-order (and higher) terms, the linear approximation first overstates the return to labor, as the capital stock increases. Hence, the linearization over-predicts initial labor supply and thus initial output. It also understates the decline in the rate of return on capital, as capital accumulates, which also leads to a consumption response that is systematically higher than in the nonlinear model, where the latter also accounts for the higher-order terms.

Comparing Tables 3A, B and 5A, B, the errors, which are in comparable units, are substantially larger than in the flow (one state variable) case. Consider the case of income and the benchmark parameters ($\sigma=1, \gamma=-1.5, \eta=0.2$). The initial error of 0.76%, while still small, is nearly 14 times larger than that for the flow model, and its more gradual decline reflects the slower rate of convergence. Even the direction of impact effects (instantaneous response to the shock) is wrong in some cases, implying a qualitatively erroneous response. For example, in Fig. 1, the nonlinear model predicts an initial 0.28% downward jump in consumption in response to the government spending shock, whereas the linear model predicts a 0.05% upward jump.

Two things are apparent from Fig. 1. Although the linearized approximation appears to track the true system reasonably closely, there are significant errors that are committed at the initial stages, following the policy change. And, occurring early in the dynamic evolution means that they weigh heavily in the welfare calculations. Fig. 1(7 and 8) illustrate the time paths for the instantaneous welfare and the accumulated welfare paths, associated with the nonlinear model and its linear approximation. On impact, the nonlinear model implies a welfare loss of 0.550% of initial consumption, whereas the linear model implies a smaller 0.313% loss.²² This is an error of 0.237 percentage points that is more or less sustained throughout the entire time path, since overall the linearized model over-predicts the long-run intertemporal gain by 0.265 percentage points (1.048% vs. 0.783%). This is significantly larger than are the corresponding errors in the flow model, both measured in percentage point terms, and as percentages of the changes themselves.

Tables 5A and B summarizes the errors from linearization for other combinations of the structural parameters, σ and γ . These are all substantially higher than are the corresponding errors for the flow model. As σ increases and γ decreases, the errors from linearization increase in magnitude, since the underlying model becomes more nonlinear in these cases. In addition, sensitivity analysis conducted for elasticity of substitution in production and elasticity of intertemporal substitution [but not reported] show that

- (1) As in the flow case, a higher elasticity of substitution reduces the speed of convergence whereas higher elasticity of intertemporal substitution increases it. Further, higher speed of convergence is typically associated with larger initial errors that die down faster when compared to the case with lower speed of convergence.
- (2) However, errors in key variables are much larger than that in the flow case, irrespective of the values of σ and γ .

An important source of nonlinearity in the model is the externality arising from public capital accumulation, parameterized by η . As this externality increases, nonlinear effects become more important. Table 5C reports the welfare changes implied by the linearized and nonlinear solutions in the benchmark case for three values of η : 0.05, 0.2, and 0.4, along with the implied errors over different time horizons. Fig. 2 illustrates the transition dynamics of the model when

²¹ We did not provide figures for the “flow” model, since the time paths track those of the nonlinear model very closely.

²² The over-estimate of consumption together with the under-estimation of leisure in the linearized model are partially offsetting in the assessment of the error in estimating welfare.

Table 5
The “Stock” model (two state variable case).

Moderate increase in g: 0.04–0.05												
A. Errors at different points of time (T)												
T	$\gamma=0$						$\gamma=-1.5$					
	0	2	5	10	20	50	0	2	5	10	20	50
$\sigma=0.50$	$\lambda_1=-0.315; \lambda_2=-0.028$						$\lambda_1=-0.150; \lambda_2=-0.027$					
C	0.245	0.398	0.459	0.411	0.260	0.061	0.333	0.389	0.423	0.407	0.296	0.079
I	0.149	0.068	0.013	-0.013	-0.016	-0.012	0.057	0.047	0.035	0.022	0.007	-0.009
K	0.000	0.316	0.485	0.475	0.307	0.072	0.000	0.212	0.397	0.495	0.413	0.119
Y	0.548	0.523	0.468	0.370	0.223	0.049	0.757	0.728	0.675	0.577	0.399	0.124
$\sigma=1.00$	$\lambda_1=-0.147; \lambda_2=-0.027$						$\lambda_1=-0.069; \lambda_2=-0.025$					
C	0.245	0.328	0.390	0.400	0.308	0.089	0.332	0.353	0.369	0.370	0.320	0.142
I	0.241	0.157	0.073	0.003	-0.030	-0.014	0.143	0.114	0.080	0.042	0.005	-0.001
K	0.000	0.198	0.372	0.468	0.396	0.119	0.000	0.130	0.268	0.396	0.447	0.238
Y	0.573	0.506	0.422	0.312	0.171	0.025	0.760	0.706	0.633	0.528	0.365	0.122
$\sigma=1.25$	$\lambda_1=-0.099; \lambda_2=-0.026$						$\lambda_1=-0.050; \lambda_2=-0.021$					
C	0.221	0.283	0.339	0.368	0.319	0.118	0.305	0.317	0.328	0.331	0.305	0.175
I	0.283	0.202	0.113	0.025	-0.036	-0.024	0.178	0.144	0.103	0.054	0.002	-0.022
K	0.000	0.146	0.292	0.407	0.407	0.161	0.000	0.091	0.195	0.306	0.386	0.281
Y	0.518	0.453	0.369	0.261	0.123	-0.004	0.673	0.619	0.546	0.444	0.295	0.085
B. Welfare gain on impact and in the long run (% of initial consumption)												
	$\gamma=0$				$\gamma=-1.5$							
	T=0		T=∞		T=0		T=∞					
$\sigma=0.50$												
Linear	-1.034		1.485		-0.475		1.393					
Nonlinear	-1.168		1.218		-0.765		1.121					
%pt error	0.134		0.267		0.290		0.272					
% Error	11.47		21.92		37.91		24.26					
$\sigma=1.00$												
Linear	-0.839		1.201		-0.313		1.048					
Nonlinear	-0.925		0.944		-0.550		0.783					
%pt error	0.086		0.257		0.237		0.265					
% Error	9.30		27.22		43.09		33.84					
$\sigma=1.25$												
Linear	-0.850		0.943		-0.351		0.744					
Nonlinear	-0.904		0.696		-0.550		0.493					
%pt error	0.054		0.247		0.199		0.251					
% Error	5.97		35.49		36.18		50.91					
C. Sensitivity analysis for welfare gain												
$\sigma=1; \gamma=0$												
$T=0$												
$T=\infty$												
$\eta=0.05$												
Linear	-1.080											
Nonlinear	-1.124											
%pt error	0.044											
% Error	3.91											
$\eta=0.20$												
Linear	-0.313											
Nonlinear	-0.550											
%pt error	0.237											
% Error	43.09											
$\eta=0.40$												
Linear	0.983											
Nonlinear	0.224											
%pt error	0.759											
% Error	338.8											

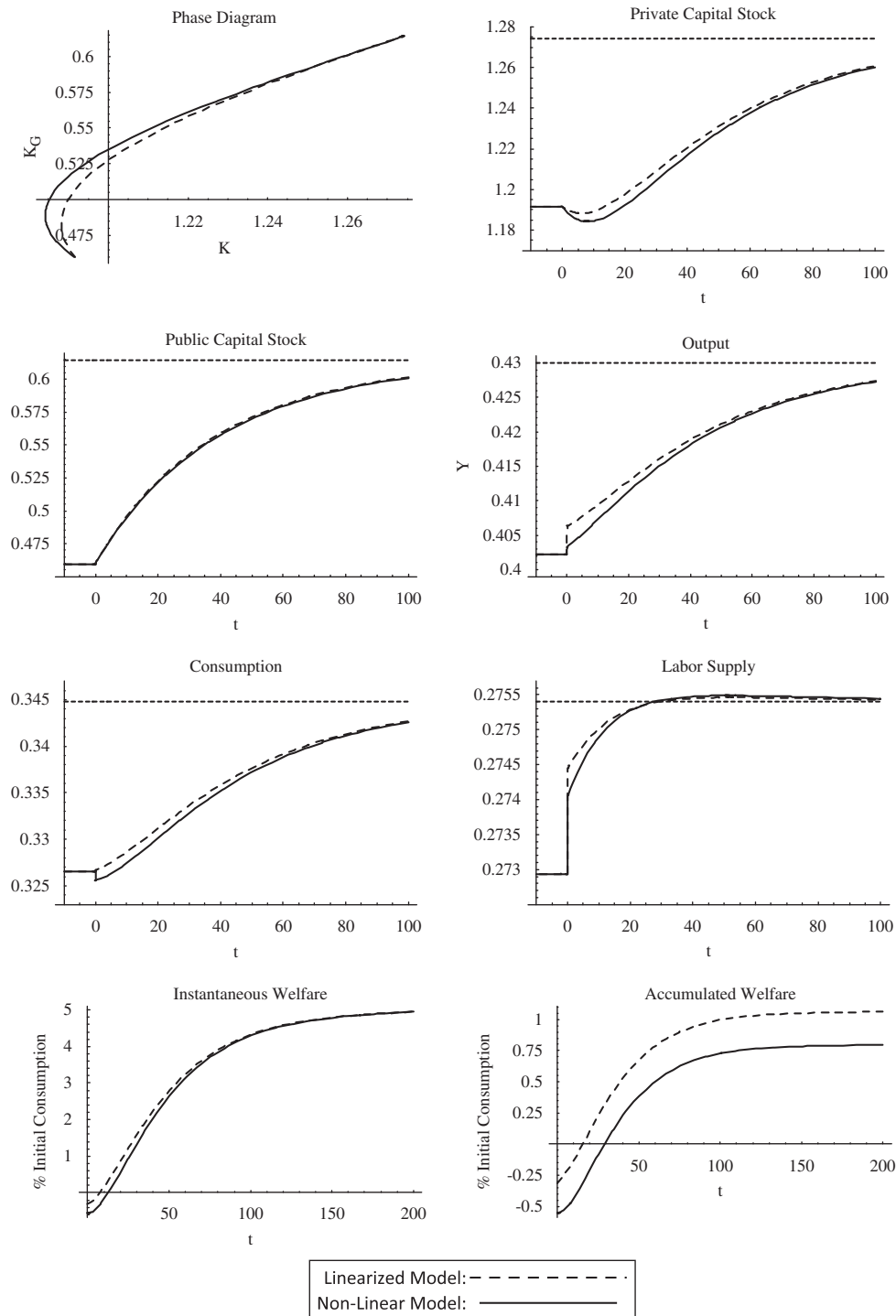


Fig. 1. The “Stock” model—a moderate increase in g : 0.04–0.05. Time paths for variables for benchmark case ($\gamma = -1.5$, $\sigma = 1$, $\eta = 0.2$).

$\eta = 0.4$. Compared to the benchmark case ($\eta = 0.2$) in Fig. 1, it can easily be seen that when the externality is large, the errors from linearization last longer during transition for all the key variables. In particular, now there are perceptible errors even in the path of public capital unlike in the benchmark case, when it was tracked closely. More significantly, the linearized approximation now yields a qualitatively *incorrect* prediction about the impact effect of the government expansion on labor supply. While the true model implies an instantaneous *downward* jump in labor supply, the linear approximation now implies an *upward* jump. This error arises because the linear approximation also ignores terms such as $F_{LK_G K_G} < 0$, which now become important and reinforces the overstatement of the real wage rate discussed in Eq. (19b). These contrasting predictions have important implications for the calculation of welfare changes. As Table 5C indicates, the welfare prediction of the linearized model deteriorates with an increase in η . For example, when $\eta = 0.4$, the percentage point errors in welfare increase by a factor of 3 in the short run and by almost 4 in the long run.

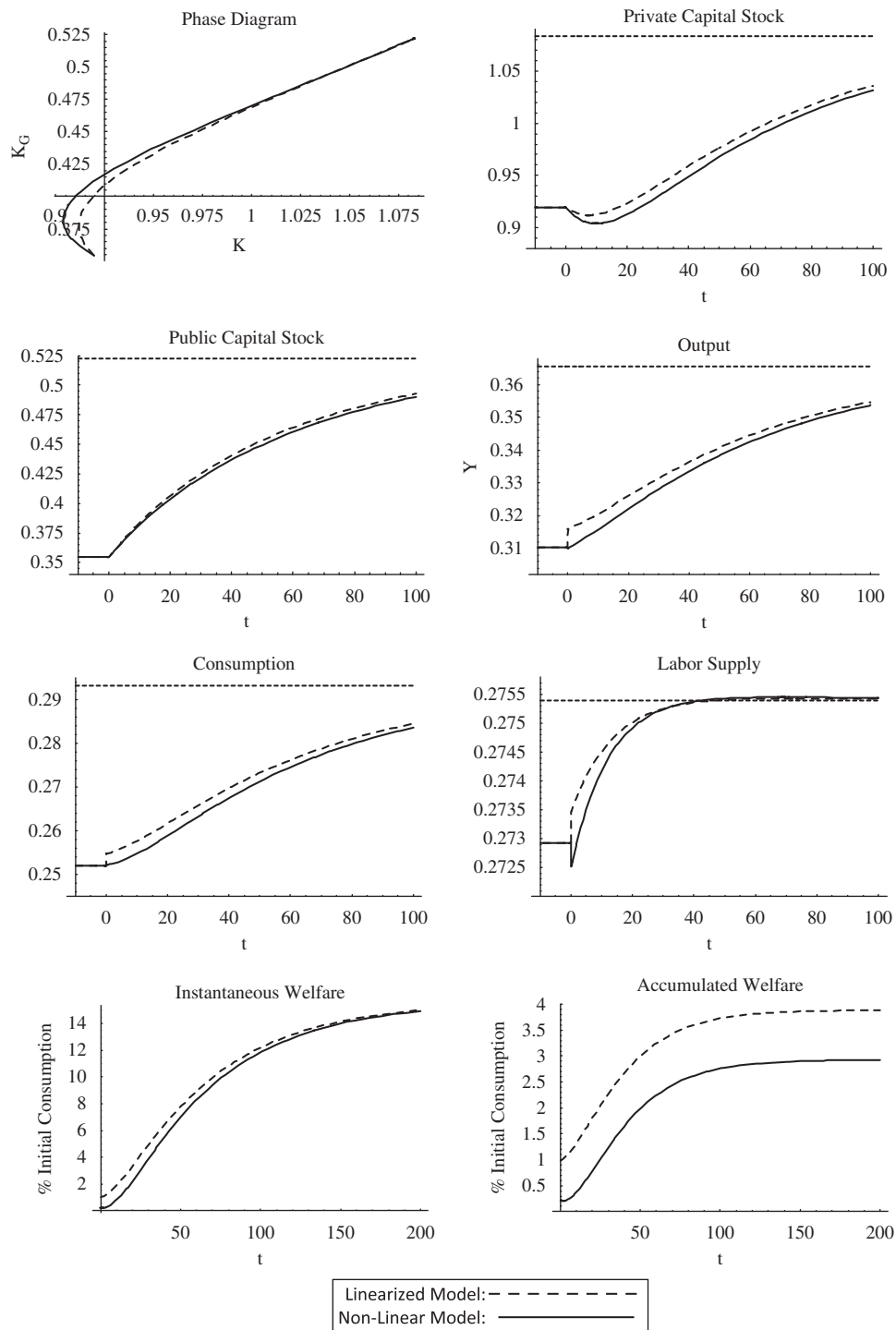


Fig. 2. The “Stock” model—a moderate increase in g : 0.04–0.05. Time path for variables when η is large ($\gamma = -1.5$, $\sigma = 1$, $\eta = 0.4$).

5.2.2. Large increase in government spending

The errors from linearization again get magnified for a large increase in g , from 0.04 to 0.08 (a 100% increase). Comparing Table 6A with 5A, they increase by a factor of around 8 in the short run which is slightly less than for the flow version. Fig. 3 illustrates the adjustment paths of the key variables from which the following conclusions, robust to variations in σ and γ , emerge:

- (1) A comparison with Fig. 2 (a moderate increase in g) reveals that errors are much larger in this case. Once again, the linearized approximation over-predicts the short-run and transitional response of all variables (consumption, labor supply, public and private capital, and output). The errors are more persistent and last for a longer time during the transition.

Table 6

The “Stock” model (two state variable case).

Large increase in g: 0.04–0.08												
A. Errors at different points of time (T)												
T	$\gamma=0$						$\gamma=-1.5$					
	0	2	5	10	20	50	0	2	5	10	20	50
$\sigma=0.50$	$\lambda_1 = -0.315; \lambda_2 = -0.028$						$\lambda_1 = -0.150; \lambda_2 = -0.027$					
C	2.15	3.49	3.99	3.51	2.17	0.559	2.97	3.44	3.71	3.53	2.52	0.736
l	1.34	0.636	0.167	-0.041	-0.053	-0.015	0.642	0.522	0.401	0.275	0.142	0.024
K	0.000	2.90	4.38	4.21	2.65	0.670	0.000	1.96	3.59	4.39	3.59	1.08
Y	4.79	4.69	4.23	3.34	2.01	0.518	5.61	5.39	4.97	4.19	2.81	0.854
$\sigma=1.00$	$\lambda_1 = -0.147; \lambda_2 = -0.027$						$\lambda_1 = -0.069; \lambda_2 = -0.025$					
C	2.12	2.84	3.36	3.40	2.54	0.745	2.91	3.07	3.19	3.16	2.69	1.19
l	2.07	1.30	0.575	0.003	-0.225	-0.071	1.29	1.00	0.680	0.341	0.050	-0.043
K	0.000	1.82	3.36	4.14	3.42	1.02	0.000	1.15	2.37	3.45	3.82	2.02
Y	4.43	4.23	3.86	3.19	2.05	0.533	5.12	4.85	4.48	3.91	2.92	1.14
$\sigma=1.25$	$\lambda_1 = -0.099; \lambda_2 = -0.026$						$\lambda_1 = -0.050; \lambda_2 = -0.021$					
C	1.88	2.43	2.90	3.13	2.66	0.967	2.68	2.77	2.84	2.85	2.59	1.48
l	2.39	1.64	0.854	0.121	-0.322	0.163	1.57	1.23	0.843	0.412	0.004	-0.142
K	0.000	1.32	2.62	3.59	3.52	1.36	0.000	0.820	1.74	2.68	3.32	2.41
Y	3.87	3.70	3.42	2.92	2.01	.571	4.46	4.21	3.89	3.43	2.69	1.27
B. Welfare gain on impact and in the long run (% of initial consumption)												
	$\gamma=0$				$\gamma=-1.5$							
	T=0	T=∞			T=0	T=∞						
$\sigma=0.50$												
Linear		-3.686		5.875		-0.834		5.498				
Nonlinear		-4.816		3.665		-3.282		3.176				
%pt error		1.130		2.210		2.448		2.322				
% Error		23.46		60.30		74.59		73.11				
$\sigma=1.00$												
Linear		-3.215		4.570		-0.538		4.188				
Nonlinear		-3.948		2.606		-2.553		1.939				
%pt error		0.733		1.964		2.015		2.249				
% Error		18.57		75.36		78.93		115.99				
$\sigma=1.25$												
Linear		-3.449		3.738		-0.883		3.057				
Nonlinear		-3.915		1.676		-2.608		0.913				
%pt error		0.466		2.062		1.725		2.144				
% Error		11.90		123.0		-66.14		243.8				
C. Sensitivity analysis for welfare gain												
$\sigma=1; \gamma=-1.5$												
	T=0				T=∞							
$\eta=0.05$												
Linear				-4.272				-2.582				
Nonlinear				-4.621				-2.939				
%pt error				0.349				0.357				
% Error				7.55				-12.15				
$\eta=0.20$												
Linear				-0.538				4.188				
Nonlinear				-2.553				1.939				
%pt error				2.015				2.249				
% Error				78.93				115.99				
$\eta=0.40$												
Linear				7.645				18.638				
Nonlinear				0.221				9.394				
%pt error				7.424				9.244				
% Error				3359				98.40				

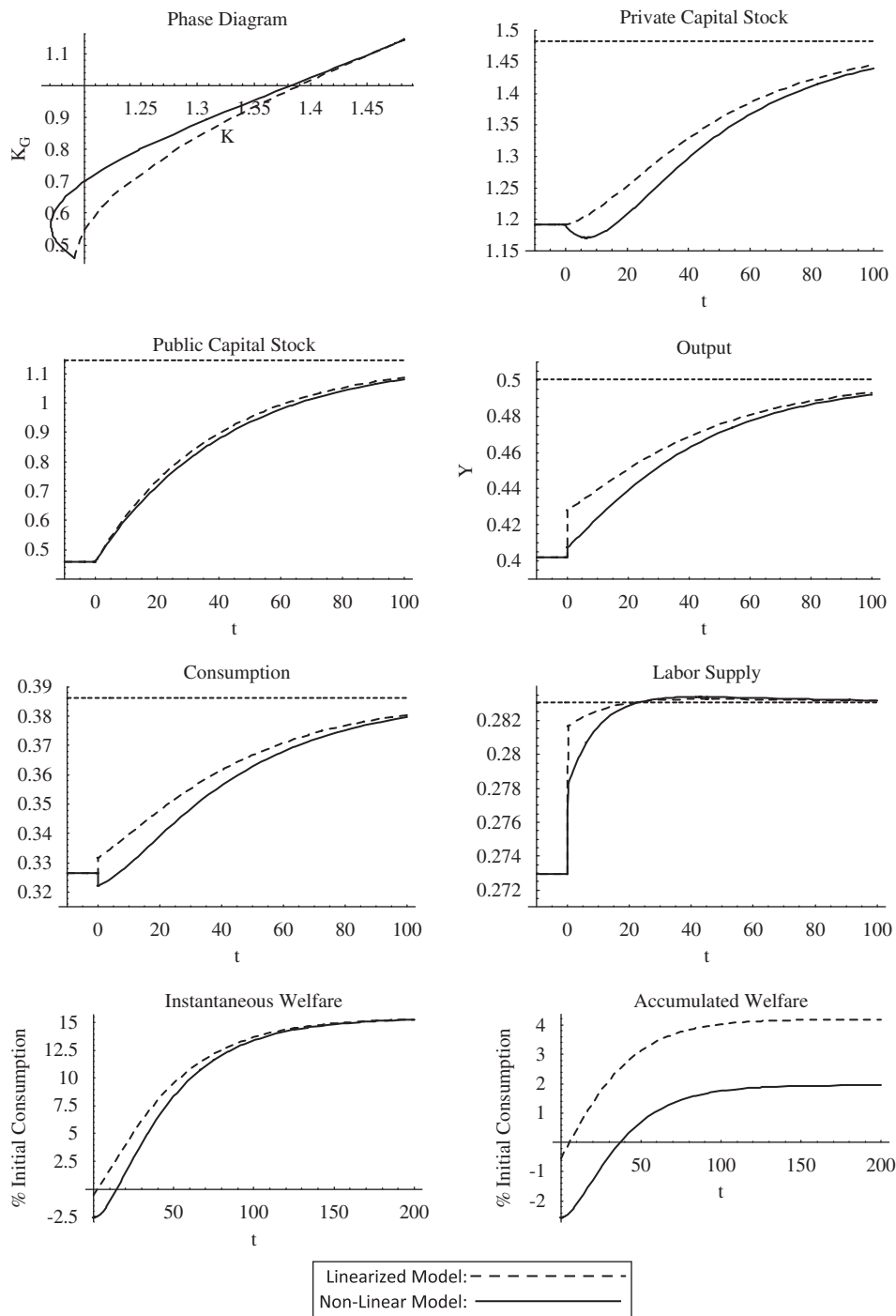


Fig. 3. The “Stock” model—a large increase in g : 0.04–0.08: $\gamma = -1.5$, $\sigma = 1$, $\eta = 0.2$.

(2) The qualitative instantaneous impact of the shock on consumption as predicted by the linearized model is misleading, when compared with the response in the nonlinear model. While the linearized model predicts an upward jump in consumption in response to the fiscal shock, the nonlinear model actually implies that consumption falls on impact. Further, the linearized model fails to predict a sharp capital decumulation in the short run in response to the strong consumption smoothing motive.

The above errors generated by the linearized model along the transition path are also reflected in errors in the calculation of the accumulated welfare gains over different time horizons, as shown in Tables 6A and B. In percentage terms, the errors are much larger than those in Table 5. In the short run, the linear approximation grossly understates the short-run losses and seriously overstates the long-run welfare gains. For example, whereas the true model predicts a long-run welfare increase of 1.939% the linear approximation overstates this by 2.249 percentage points. Looking at

Fig. 1(8) we see that for between period 6 and 38 the linear approximation will suggest positive accumulated welfare gains, whereas in fact the true nonlinear model indicates accumulated losses during that time frame. In some cases the errors are seriously misleading. For example, the linear approximation suggests that doubling the rate of government investment will lead to accumulated welfare gains of just over 1% over the first 20 years, whereas in fact the true model suggests an almost 1.4% welfare loss. In absolute terms, the errors in welfare predictions of the linearized model hover around 2% of initial consumption, in contrast to more modest errors of 0.2–0.3% for the moderate increase case. These are huge errors given the fact that the policy generates welfare gains equivalent to 1–4% of initial consumption.

Finally, we may note that the effects of the higher externality associated with public capital, η , illustrated in Fig. 2 for the moderate expenditure increase apply here as well. As in Fig. 2, the errors from linearization persist longer during transition for all the key variables, including the path of public capital, and the impact effect on labor supply is in the wrong direction (relative to the nonlinear model). The effect of the variation in η on the calculation of intertemporal welfare is reported in Table 6C. As in Table 5C, an increase in η increases the absolute error from linearization in accumulated welfare changes over all time horizons. However, now the errors are quantitatively much larger and more persistent. For the high value of $\eta=0.40$, the linearized solution over-predicts accumulated welfare at different time horizons by 8–9% of initial consumption, compared to the 2–2.5% over-prediction for the case where $\eta=0.2$. Even for $\eta=0.05$, the errors are quite significant, though much smaller. Interestingly, for such a low η , a large increase in g reduces overall welfare as it takes public investment away from its optimal level.

5.3. Dimensionality and adjustment speed

Comparing the numerical results in Sections 5.1 and 5.2, it appears that the higher-order stock model is associated with substantially larger errors than is the flow model. To test whether this is due to the dimensionality of the system or the fact that the stock system has a larger (stable) eigenvalue generating a slower asymptotic adjustment, we run additional simulations. To accelerate the adjustment we increase the rate of depreciation of public capital in the stock model from 3.5% to 21%. The resulting simulations (available on request) show indeed that errors from linearization are, substantially reduced with the more rapid adjustment, and the welfare calculations also show much smaller errors.²³

Thus we conclude that the critical factor is the slower adjustment, (rather than the dimensionality per se), as it keeps the economy away from the final steady state for a longer time. And it is during this period (when economy is far from steady state) that the linearization around the final steady state is likely to result in larger errors. At the same time, the fact that slower adjustment leads to protracted errors in the path of various variables and in the assessment of welfare effects, has to be interpreted in the context of a particular structure. What we establish is that in the context of a particular model (neoclassical model with public spending) the characteristics that slow down the adjustment process would lead to larger errors from approximation due to linearization. While it is not possible to make comparisons across models about the order of the system and resulting errors, our insight is valid across similar models looking to address a particular issue.

6. Temporary shocks

While a permanent increase in government spending serves as the natural benchmark illustration of the potential errors from linearization, governments frequently increase investment as a temporary source of stimulus. It is therefore also important to consider the sensitivity of these errors to the *duration* of a shock. In this section we briefly summarize the effect of temporary fiscal shocks on the predictions of the linearized and nonlinear solutions to the models developed in Sections 2 and 3. We consider both moderate and large spending shocks as in the previous sections, but assume that these shocks now last for a 5-year period, after which government spending is restored to its initial, pre-shock level.

6.1. Temporary shocks in the “flow” model

As in the case of a permanent shock, the errors from linearizing the flow model with public investment in the benchmark case ($\sigma=1$) are minimal, both for the moderate and the large government spending shock. This leads us to infer that the duration of a policy shock does not generate serious errors from linearization in the flow model with a Cobb–Douglas technology. However, when the elasticity of substitution in production increases from $\sigma=1$ to 1.25, the errors from linearization start increasing. One consequence of the higher elasticity of substitution is that it slows down the speed of convergence to the pre-shock steady-state (as the temporary shock ends, the economy must transition back to its pre-shock saddle path). As discussed in Section 5, a slower speed of convergence can generate larger errors from linearization. Overall, our experiments suggest that the flow model does a reasonable job of approximating the “true” dynamic path of a system, irrespective of whether the shock is permanent or temporary, although if the elasticity of substitution increases above unity, the errors start to increase.²⁴

²³ As evidence for this we see that the two stable eigenvalues increase (in magnitude) from -0.069 and -0.025 for the benchmark model to -0.189 and -0.051 , respectively, with the more rapidly depreciating public capital.

²⁴ We do not illustrate these results graphically, but the figures are available from the authors on request.

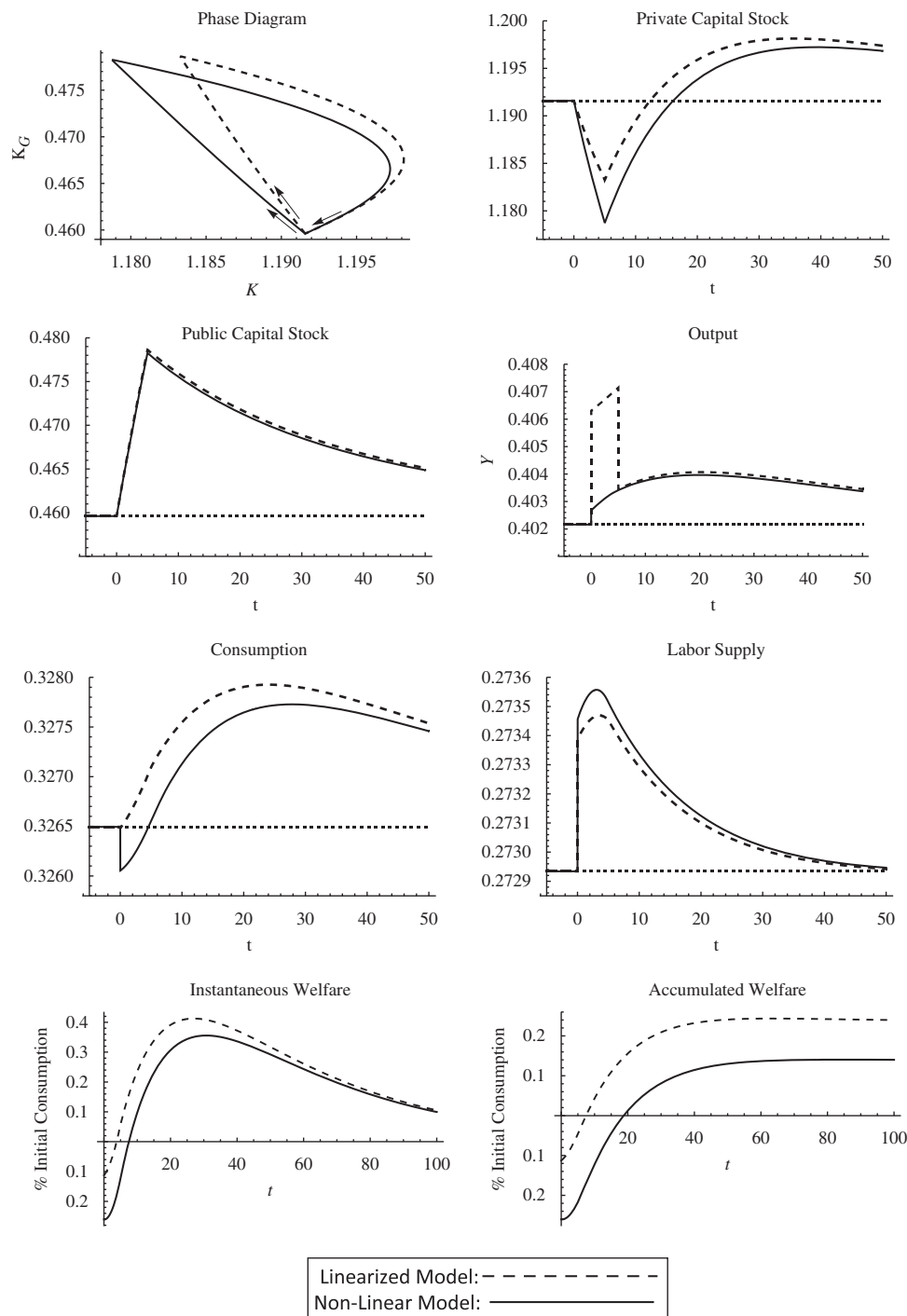


Fig. 4. The “Stock” model—a moderate temporary 5 year increase in g : 0.04–0.05. Time paths for variables for benchmark case ($\gamma = -1.5$, $\sigma = 1$, $\eta = 0.2$).

In the case of a permanent shock, we have noted previously that the linearized model, in some cases, incorrectly predicted the instantaneous responses of consumption and leisure, which eventually contributed to an over-prediction of intertemporal welfare changes across steady states. However, in the case of a temporary shock two differences arise. First, the *direction* of the instantaneous responses are correctly predicted by the linearized model. Second, instead of over-predicting consumption on impact, linearization now over-predicts consumption at the time of withdrawal of the shock. This is due to the fact while the economy jumps to the stable manifold on impact for the permanent case; it does so for the temporary case only at the time of the withdrawal of the shock after following the unstable path for the duration of the shock.

6.2. Temporary shocks in the “stock” model

As demonstrated in Section 5, introducing public capital as a stock can lead to both qualitative and quantitative errors from linearization that are substantially larger than those in the flow model. This pattern is magnified in the case of a

temporary shock, as shown in Fig. 4. Even with a moderate temporary shock, the errors from linearization are larger than for the corresponding permanent shock. This is because the introduction of a second state variable (public capital) slows down the speed of the transition path, both when the shock is in effect (along the unstable trajectory) and when it has ended (along the stable trajectory). This leads to a compounding of the errors. Whereas linearization committed qualitative errors in assessing the short-run response of consumption only for large permanent shocks, this occurs even for moderate temporary shocks.

Comparing the linearized and nonlinear solutions for the stock model with temporary and permanent shocks, one interesting difference arises. While the linearized model still over-predicts consumption, in the temporary case, labor supply is underpredicted in contrast to the permanent case. As a result, the impacts of errors in consumption and leisure on welfare are mutually reinforcing for a temporary shock leading to larger errors in the assessment of welfare.

7. Speed of convergence

As one example of where linearization can lead to seriously misleading qualitative implications we briefly consider the question of the speed of convergence. This describes the speed at which a dynamic system approaches its steady-state equilibrium and is an important practical aspect in the evaluation of the impact of public policies. For example, if convergence to the steady state is rapid, then one can evaluate public policies with respect to their long-run impact on the economy. On the other hand, if convergence is slow, then a significant part of the dynamic adjustment takes place far away from the steady-state, which in turn underscores the need for the study of the transitional aspects of public policies. In this context, a natural question that arises is: do *linearized* systems over-predict or under-predict the speed of convergence relative to those implied by *nonlinear* systems? This is a significant question, since the implied speeds of convergence from simple one-sector models of growth, based on approximate linearized dynamics, are typically much larger than those estimated from the data. This discrepancy, as we illustrate below, can be mitigated to a large extent if estimates are based on exact nonlinear dynamics. Thus, the paper provides some insights into how theory might be reconciled with facts: perhaps an important factor is the manner in which the dynamics of theoretical growth models are approximated or analyzed.

Following Barro and Sala-i-Martin (1992), we consider the following measure of the speed of the transitional path, called β -convergence²⁵:

$$v_x(t) = -\frac{\ln[\dot{x}(t)/x(t)]}{\ln[x(t)]}$$

where $x(t)$ is a generic state variable, such as the capital stock. The speed of convergence v_x therefore measures the rate at which the growth of the variable x slows down as it increases proportionately (i.e., approaches its steady-state equilibrium). For our analysis, this speed of convergence is calculated and compared for both the linearized and nonlinear models in the cases where government expenditure enters production as a flow or a stock.

Fig. 5 depicts the speed of convergence implied by the linearized and nonlinear models for both the flow and stock cases and its sensitivity to the magnitude of policy changes and structural parameters. The following patterns emerge from this comparison:

- (1) In the flow model (Panel I), the speed of convergence of the linearized transition path is always higher than that implied by the nonlinear path. However, the magnitude of this over-prediction is quite small, and the two measures converge quite rapidly following a particular shock. Thus, consistent with our previous analysis, the errors from linearization do not seem to be too serious when the dynamic system is characterized by a quick transitional adjustment (e.g. one state variable).
- (2) In the stock model, the errors in the calculation of the speed of convergence from linearization are much more significant, both qualitatively and quantitatively. First of all, the initial jump in the speed of convergence on the impact of a shock is substantially larger in the linearized model relative to the nonlinear model. Second, in three out of the four shocks considered (a large increase in g , a fall in σ , and an increase in γ), the short-run impact on the speed of convergence is exactly the *opposite* in the linearized and nonlinear models: while the linearized path implies an instantaneous increase in the speed of convergence, the nonlinear path implies an instantaneous decline. Third, during transition, the speed of convergence implied by the linearized adjustment path is consistently higher than

²⁵ Several measures of the speed of convergence can be found in the literature. A frequently used metric is the stable eigenvalue of the linearized system, as in Ortiguera and Santos (1997). While this measure is suitable for one-dimensional transition paths, it is not appropriate for systems with more than one state variable. Another measure, proposed by Eicher and Turnovsky (1999), characterizes the time-varying nature of the speed of convergence in higher-order models, and is defined as: $v_x(t) = -[(\dot{x}(t) - \dot{\bar{x}})/(x(t) - \bar{x})]$. This measure is quantitatively similar to the β -convergence measure we use for our analysis.

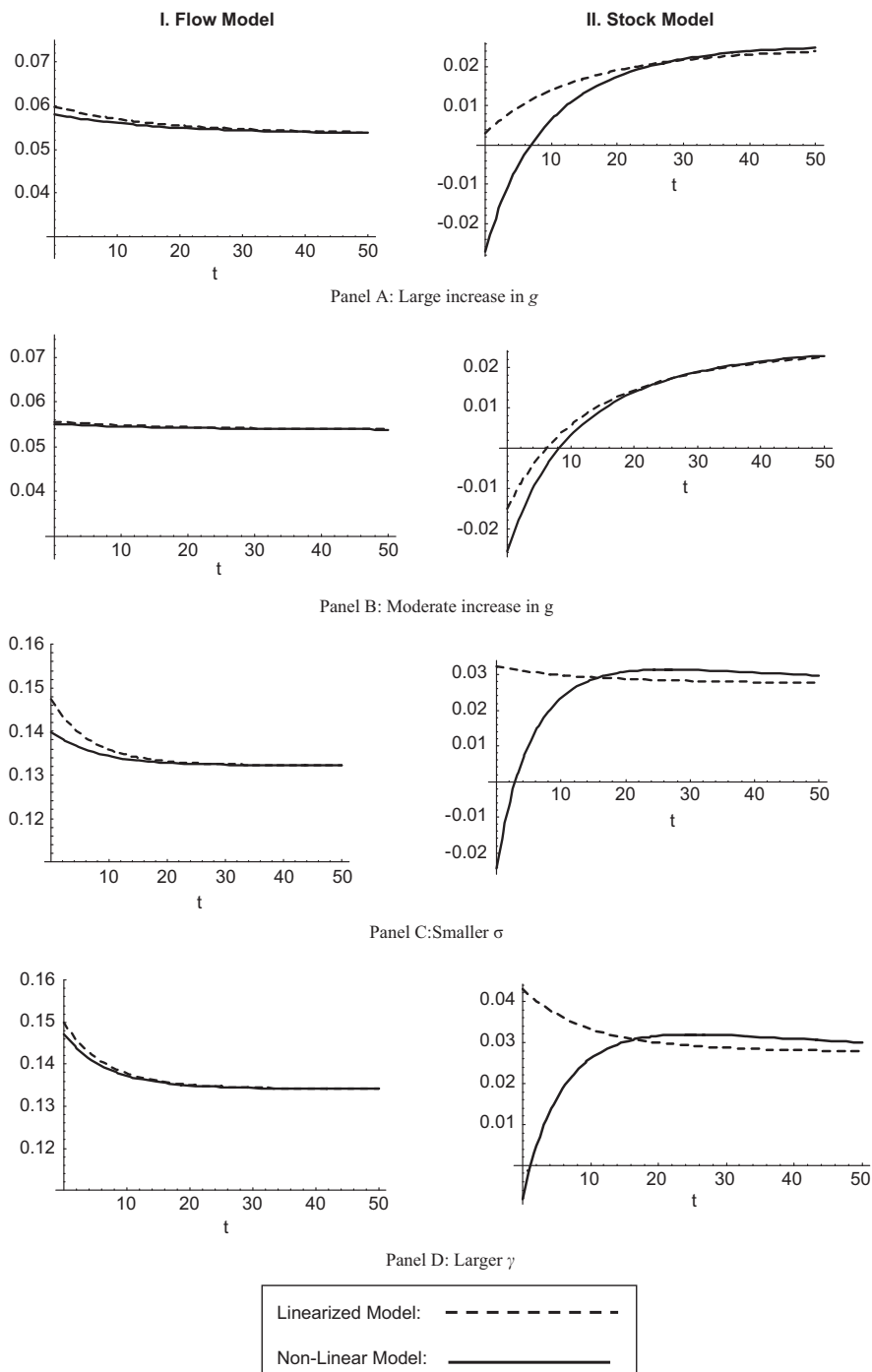


Fig. 5. Speed of convergence in the “Flow” and “Stock” models.

that implied by the nonlinear path. This difference lasts for an average of 15–20 years following each shock, after which there is a tendency for both measures to converge.²⁶

8. Conclusions

In this paper we have addressed an important practical question with respect to the effects of government policy on the dynamic evolution of an economy: how serious are the errors in employing conventional linearized solution methods? To

²⁶ As mentioned in the introduction, Papageorgiou and Perez-Sebastian (2007) show that the asymptotic speed of convergence implied by a linearized growth model does a poor job of replicating the actual convergence paths of countries like Japan and South Korea. Their analysis, therefore, calls for a complete characterization of the transition path, rather than just a study of the asymptotic dynamics (implied by linearization). Our comparison of the linearized and nonlinear convergence paths is thus consistent with their findings.

an important degree, the answer to this question depends upon the policymaker's tolerance for error. To address this, we have considered two versions of a Ramsey-type growth model. In the first, government expenditure appears as a flow and the economy is described by a first-order dynamic system, whereas in the second, it is introduced as a stock, thereby rendering the dynamics to be of second order.

Overall, our results suggest that in the flow model, the errors committed by linearization are small and do not generate seriously misleading results. Even if one doubles the rate of government spending – a large quantitative change – the errors from linearization are tolerable. In the short run they are typically less than 1% of the initial value of the respective variable and they decline rapidly over time. The corresponding errors for welfare gains (or losses) are also small, being less than one percentage point in the short run to around 0.2–0.3 percentage points in terms of the long-run accumulated welfare gains. Given the model's parsimony, this would seem to be a tolerable error.

The stock specification is more difficult. The errors generally are of the order of 6 times those of the flow model. This is because the higher-order system generates slower adjustments and these are associated with larger errors. In the short run, the linear approximation may yield incorrect *qualitative* predictions. This is particularly true as the importance of government capital in production increases, and the size of the change in government investment increases. In these cases the linearized model can yield misleading conclusions insofar as the short run is concerned. These qualitative errors have a critical impact on intertemporal welfare calculations: the error in estimating welfare changes caused by a change in g is 33.8% in the benchmark stock case for a moderate shock, and rises to 116% when the shock is large. For some experiments, the errors were as high as 244%.

We have also examined whether the source of the errors from linearization lies in the dimensionality of the system (i.e. the number of state variables) or the speed of adjustment to the steady-state equilibrium. Our numerical experiments indicate that the latter rather than the former is the more critical factor. Finally, we consider the case of temporary shocks, and find that the errors from linearization are generally more magnified relative to those generated by permanent shocks.

However, provided one proceeds judiciously, our analysis suggests that linearization may still be useful. In this regard, if: (i) one restricts oneself to “moderate policy changes” as in Table 5 and (ii) focuses on the longer run when the linear model tracks the true path more closely, the errors are not seriously misleading. Thus, if one uses the model to provide a “bottom line” estimate of the long-run benefits of the policy change, errors of the order summarized in Table 5B should be tolerable. But with the higher-order model, linearization may be less reliable for short-run analysis, when the errors can be substantial, both quantitatively and qualitatively. In that case, one may employ nonlinear solution procedures such as forward and reverse shooting, the application of which has become more practical with the rapid increase in computing capacity.

We conclude our discussion with two observations. First, while our focus has been on linearization, which is the most widely practiced form of approximation, to check the robustness of our results we have conducted some simulations based on log-linearization of the model. Here we find the performance to be mixed. For example, the errors in labor/leisure decline, while consumption shows smaller errors on impact, but larger errors later in the transition. In contrast, public capital and output generate uniformly larger errors. The comparison generally supports those of Aruoba et al. (2006) who, in comparing various solution methods for DSGE models, find linearization to be superior to log-linearization. But as in other aspects of our study, the comparison depends upon the underlying context, and care must be exercised before drawing sweeping conclusions.

It is also important to emphasize that errors in approximate solutions are generated by the (often complex) interaction of the endogenous and exogenous variables which depends on a variety of factors such as the underlying shock (structural versus policy), its magnitude (big, moderate, or small shocks), and structure (number of state variables, open or closed economy, etc.) of the model. While this in general precludes deriving general conclusions, our results highlight that the linear approximation of transition paths does generate errors when compared to nonlinear solutions, and we have attempted to characterize some of the factors that determine their magnitude, direction, and their significance, particularly with respect to their policy implications.

Appendix

In this paper, we report accumulated welfare gains over different time horizons, T . We quantify the welfare gains as the (equivalent) percent increase in consumption over initial consumption that delivers same utility over lifetime as the increase in g .

To compute this measure over time horizon T , we first compute the total utility over the time horizon after implementation of the increase in g

$$W_T(C(t), l(t)) = \int_0^T \frac{1}{\gamma} [C(t)l(t)^\theta]^\gamma e^{-\beta t} dt$$

where $C(t)$ and $l(t)$ are time paths of consumption and leisure based on the relevant model, linear or nonlinear. The corresponding total utility in absence of increase in g is

$$W_T(C_0, l_0) = \int_0^T \frac{1}{\gamma} [C_0 l_0^\theta]^\gamma e^{-\beta t} dt$$

We seek to find ζ such that

$$W_T(\zeta C_0, l_0) = W_T(C(t), l(t))$$

which implies that the (equivalent) percent increase in consumption over initial consumption as reported in Panels B of Tables 3 and 4

$$\zeta - 1 = \left[\frac{W_T(C(t), l(t))}{W_T(C_0, l_0)} \right]^{1/\gamma} - 1$$

It may be mentioned that this consumption equivalent measure of welfare change takes into account the utility gain or loss arising out of change in leisure due to increase in g .

References

- Agenor, P.R., Montiel, P.J., 1996. In: *Development Macroeconomics*. Princeton University Press, Princeton, NJ.
- Antràs, P., 2004. Is the U.S. aggregate production function Cobb–Douglas? New estimates of the elasticity of substitution. *Contributions to Macroeconomics* 4, 1–34.
- Arrow, K.J.M., Kurz, M., 1970. In: *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. Johns Hopkins Press, Baltimore, MD.
- Aruoba, S.B., Fernandez-Villaverde, J., Rubio-Ramirez, J.F., 2006. Comparing solution methods for dynamic equilibrium economies. *Journal of Economic Dynamics and Control* 30, 2477–2508.
- Atolia, M., Buffie, E.F., 2009a. Smart forward shooting. *Computational Economics* 33, 1–30.
- Atolia, M., Buffie, E.F., 2009b. Reverse shooting made easy: automating the search for the global nonlinear saddle path. *Computational Economics* 34, 273–308.
- Azariadis, C., 1993. In: *Intertemporal Macroeconomics*. Blackwell, Oxford.
- Barro, R.J., 1990. Government spending in a simple model of endogenous growth. *Journal of Political Economy* 98, S103–S125.
- Barro, R.J., Sala-i-Martin, X., 1992. Convergence. *Journal of Political Economy* 100, 223–251.
- Baxter, M., King, R., 1993. Fiscal policy in general equilibrium. *American Economic Review* 83, 315–334.
- Becker, S., Grüne, L., Semmler, W., 2007. Comparing accuracy of second-order approximation and dynamic programming. *Computational Economics* 30, 65–91.
- Berndt, E.R., 1976. Reconciling alternative estimates of the elasticity of substitution. *Review of Economics and Statistics* 58, 59–68.
- Blanchard, O.J., Fischer, S., 1989. In: *Lectures on Macroeconomics*. MIT Press, Cambridge MA.
- Brunner, M., Strulik, H., 2002. Solution of perfect foresight saddlepoint problems: a simple method and applications. *Journal of Economic Dynamics and Control* 26, 737–753.
- Caselli, F., Esquivel, G., Lefort, F., 1996. Reopening the convergence debate: a new look at cross-country empirics. *Journal of Economic Growth* 1, 363–390.
- Chatterjee, S., 2005. Capital utilization, economic growth, and convergence. *Journal of Economic Dynamics and Control* 29, 2093–2124.
- Cooley, T.F. (Ed.), 1995. *Frontiers of Business Cycle Research*. Princeton University Press, Princeton, NJ.
- Den Haan, W.J., Marcat, A., 1994. Accuracy in simulations. *Review of Economic Studies* 61, 3–17.
- Eicher, T., Turnovsky, S.J., 1999. Convergence speeds and transition dynamics in non-scale growth models. *Journal of Economic Growth* 4, 413–428.
- Evans, P., 1997. How fast do countries converge? *Review of Economics and Statistics* 79, 219–225.
- Fisher, W.H., Turnovsky, S.J., 1998. Public investment, congestion, and private capital accumulation. *Economic Journal* 108, 399–413.
- Gramlich, E.M., 1994. Infrastructure investment: a review essay. *Journal of Economic Literature* 32, 1176–1196.
- Judd, K.L., 1992. Projection methods for solving aggregate growth models. *Journal of Economic Theory* 58, 410–452.
- Judd, K.L., 1998. In: *Numerical Methods in Economics*. MIT Press, Cambridge MA.
- Islam, N., 1995. Growth empirics: a panel data approach. *Quarterly Journal of Economics* 110, 1127–1170.
- Karakitsos, E., Rustem, B., 1989. The implications of assuming that the world is flat. In: Christodoulakis, N.M. (Ed.), *Dynamic Modelling and Control of National Economies*. Pergamon Press, Oxford, pp. 1989.
- Kim, J., Kim, S.H., 2003. Spurious welfare reversals in international business cycle models. *Journal of International Economics* 60, 471–500.
- Klump, R., McAdam, P., Willman, A., 2007. Factor substitution and factor-augmenting technical progress in the United States: a normalized supply-side system approach. *Review of Economics and Statistics* 89, 183–192.
- Mankiw, N.G., Romer, D., Weil, D.N., 1992. A contribution to the empirics of economic growth. *Quarterly Journal of Economics* 107, 407–437.
- Ortigueira, S., Santos, M.S., 1997. On convergence in endogenous growth models. *American Economic Review* 87, 383–399.
- Papageorgiou, C., Perez-Sebastian, F., 2007. Is the asymptotic speed of convergence a good proxy for the transitional growth path? *Journal of Money, Credit, and Banking* 39, 1–24.
- Rodrik, D., 1996. Understanding economic policy reform. *Journal of Economic Literature* 34, 9–41.
- Santos, M.S., 2000. Accuracy of numerical solutions using the Euler equation residuals. *Econometrica* 68, 1377–1402.
- Sargent, T.J., 1979. In: *Macroeconomic Theory*. Academic Press, NY.
- Temple, J.R.W., 1998. Robustness test of the augmented Solow model. *Journal of Applied Econometrics* 13, 361–375.
- Turnovsky, S.J., 2000. In: *Methods of Macroeconomic Dynamics* second ed MIT Press, Cambridge, MA.
- Turnovsky, S.J., 2002. Intertemporal and intratemporal substitution, and the speed of convergence in the neoclassical growth model. *Journal of Economic Dynamics and Control* 26, 1765–1785.
- Turnovsky, S.J., 2004. The transition dynamics of fiscal policy: long-run capital accumulation and growth. *Journal of Money, Credit, and Banking* 36, 883–910.