

Econ 8510/Chatterjee
Assignment 1
Solutions to Q1 and Q2

Question 1

i. We have

$$m_t - p_t = \phi y_t - \lambda i_t + \varepsilon_t \quad (1a)$$

$$m_t^* - p_t^* = \phi y_t^* - \lambda i_t^* + \varepsilon_t^* \quad (1b)$$

Subtracting (1b) from (1a) and using the assumption of PPP ($s_t = p_t - p_t^*$) and UIP ($i = i^* + E_t(s_{t+1} - s_t)$), we can derive

$$s_t = f_t + \lambda E_t(s_{t+1} - s_t) \quad (2)$$

where, f_t represents the monetary fundamental, given by

$$f_t = m_t - m_t^* - \phi(y_t - y_t^*) - (\varepsilon_t - \varepsilon_t^*) \quad (2a)$$

Re-arranging terms in (2), we get

$$s_t = \gamma f_t + \psi E_t s_{t+1} \quad (3)$$

where $\gamma = 1/(1 + \lambda)$ and $\psi = \lambda/(1 + \lambda)$.

Advance (3) by one period:

$$s_{t+1} = \gamma f_{t+1} + \psi E_{t+1} s_{t+2}$$

Taking expectations conditional on time t yields

$$E_t s_{t+1} = \gamma E_t f_{t+1} + \psi E_t s_{t+2} \quad (4)$$

Substituting (4) into (3) gives

$$s_t = \gamma f_t + \psi [\gamma E_t f_{t+1} + \psi E_t s_{t+2}] = \gamma f_t + \psi \gamma E_t f_{t+1} + \psi^2 E_t s_{t+2} \quad (5)$$

Repeat the above process for $s_{t+2}, s_{t+3}, \dots, s_{t+k}$ to get

$$s_t = \gamma \sum_{j=0}^k \psi^j E_t f_{t+j} + \psi^{k+1} E_t s_{t+k+1} \quad (6)$$

The solution will be complete by characterizing the behavior of s_t as $k \rightarrow \infty$. This requires restricting the behavior of the term $\psi^{k+1} E_t s_{t+k+1}$ as $k \rightarrow \infty$. Imposing the condition that the exchange rate be bounded in the long run (i.e., a transversality condition), we get

$$\lim_{k \rightarrow \infty} \psi^k E_t s_{t+k} = 0 \quad (7)$$

Using (7) in (6) as $k \rightarrow \infty$, gives the forward-looking solution for the nominal exchange

rate:

$$s_t = \gamma \sum_{j=0}^{\infty} \psi^j E_t f_{t+j} \quad (8)$$

Further, noting that $\psi = \lambda/(1 + \lambda) < 1$, (8) is a stable solution. It represents a "present value" expression for the exchange rate: it is the discounted present value of future expected values of the monetary fundamental.

ii. The monetary fundamental is given by

$$f_t = m_t - m_t^* - \phi(y_t - y_t^*) - (\varepsilon_t - \varepsilon_t^*)$$

From (8), we see that the nominal exchange rate is positively related to the monetary fundamental. Any shock that increases f_t will tend to depreciate the exchange rate and vice versa. For example, ceteris paribus, if domestic money supply (m_t) increases, the exchange rate will depreciate. The answer should be elaborated along these lines.

Question 2

i. The current-value Hamiltonian for the private agent's optimization problem is

$$H = u(c, l)e^{-\beta t} + \lambda e^{-\beta t} [\dot{z} - c - (1 + \tau)i(z)z + T + f(l)] \quad (1)$$

Note that the agent treats the cost of borrowing, $i(z)$, as exogenous to its allocation decisions, even though $i(z)$ will be determined endogenously from the macroeconomic equilibrium. λ is the marginal utility of wealth.

The optimality conditions are:

$$u_c(c, l) = \lambda \quad (2a)$$

$$u_l(c, l) = -\lambda f_l(l) \quad (2b)$$

$$\beta - \frac{\dot{\lambda}}{\lambda} = (1 + \tau)i(z) \quad (2c)$$

$$\lim_{t \rightarrow \infty} \lambda z e^{-\beta t} = 0 \quad (2d)$$

The optimal policy functions for c and l can be derived from (2a) and (2b):

$$c = c(\lambda), \quad c_\lambda < 0 \quad (3a)$$

$$l = l(\lambda), \quad l_\lambda > 0 \quad (3b)$$

The aggregate resource constraint (current account) for the economy can be derived by combining the private and government budget constraints:

$$\dot{z} = c + i(z)z - f(l) \quad (4)$$

The core dynamics of the economy are expressed in terms of the stock of debt, z , and the marginal utility of wealth, λ , and are given by (4) and (2c), while noting (3a) and (3b):

$$\dot{z} = c(\lambda) + i(z)z - f[l(\lambda)] \quad (5a)$$

$$\dot{\lambda} = \lambda[\beta - (1 + \tau)i(z)] \quad (5b)$$

The steady-state is stationary and is attained when $\dot{z} = \dot{\lambda} = 0$. This yields

$$f[l(\tilde{\lambda})] = c(\tilde{\lambda}) + i(\tilde{z})\tilde{z} \quad (6a)$$

$$\beta = (1 + \tau)i(\tilde{z}) \quad (6b)$$

The steady-state conditions (6a) and (6b) can be solved for \tilde{z} and $\tilde{\lambda}$. Once \tilde{z} is known, the equilibrium cost of borrowing (interest rate) is immediately determined from the upward-sloping supply curve of debt:

$$\tilde{i} = i^* + \omega(\tilde{z})$$

Finally, $\tilde{\lambda}$ determines equilibrium consumption, labor supply, and output, given (3a), (3b) and the production function.

The linearized dynamics for the system described in (5a) and (5b) is given by

$$\begin{bmatrix} \dot{z} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} i_z \tilde{z} + \tilde{i} & c_\lambda - f_l l_\lambda \\ -\lambda(1 + \tau)i_z & 0 \end{bmatrix} \begin{bmatrix} z - \tilde{z} \\ \lambda - \tilde{\lambda} \end{bmatrix} \quad (7)$$

The determinant of the coefficient matrix in (7) is

$$J = \lambda(1 + \tau)i_z(c_\lambda - f_l l_\lambda) < 0$$

since $i_z > 0$.

The system therefore displays saddlepoint dynamics and the equilibrium locus is represented by a saddle path. The optimal time paths for z and λ are

$$z(t) - \tilde{z} = (z_0 - \tilde{z})e^{-\mu t} \quad (8a)$$

$$\lambda(t) - \tilde{\lambda} = \frac{\mu - (i_z \tilde{z} + \tilde{i})}{(c_\lambda - f_l l_\lambda)} [z(t) - \tilde{z}] \quad (8b)$$

(8b) is also a representation of the saddle path in (λ, z) space and is *upward-sloping*.

ii. An increase in the rate of time preference, β .

Steady-state effects:

Differentiate (6a) and (6b) w.r.t β to derive

$$\frac{d\tilde{z}}{d\beta} = -\frac{(c_\lambda - f_l l_\lambda)}{\Delta} > 0, \quad \frac{d\tilde{\lambda}}{d\beta} = \frac{i_z \tilde{z} + \tilde{i}}{\Delta} > 0 \quad (9)$$

where $\Delta = -(1 + \tau)(c_\lambda - f_l l_\lambda)i_z > 0$.

Short-run effects:

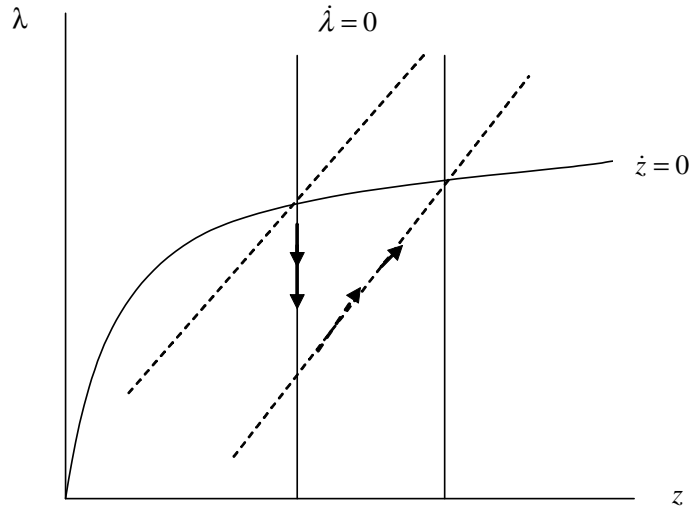
Differentiate the saddle path (8b) w.r.t β and evaluate the derivative at $t = 0$:

$$\frac{d\lambda(0)}{d\beta} = \frac{d\tilde{\lambda}}{d\beta} - \left[\frac{\mu - (i_z \tilde{z} + \tilde{i})}{(c_\lambda - f_l l_\lambda)} \right] \frac{d\tilde{z}}{d\beta} \quad (10)$$

Substituting from (9) into (10), we can easily show that

$$\frac{d\lambda(0)}{d\beta} = \frac{\mu}{\Delta} < 0$$

i.e., the instantaneous response of the marginal utility of wealth is to jump down, as shown in the phase diagram below.



The increase in β makes the representative agent more impatient by increasing the relative importance of current consumption vis-a-vis future consumption. Since the stock of debt is fixed in the short run (on impact of the shock), the only way the agent can increase current consumption is by increasing the flow of current output which, in turn, requires an instantaneous increase in labor supply; see (6a). This causes the marginal utility of wealth to jump down on incidence of the shock. Following this initial response, diminishing returns to labor reduces the increase in future output and, combined with the increase in current consumption, requires an increase in foreign borrowing (to finance the excess consumption). This raises the stock of foreign debt along the transition path. Since increasing debt implies declining private wealth, the marginal utility of wealth increases in transition. In the long run, the increase in the stock of debt and debt-financing costs implies that λ is higher than its pre-shock equilibrium. From (3a), we can then infer that long-run consumption is below its pre-shock level.

iii. *Social Planner's Problem:*

The current value Hamiltonian for the social planner is

$$H = u(c, l)e^{-\beta t} + \lambda e^{-\beta t}[\dot{z} - c - i(z)z + f(l)] \quad (11)$$

Note that (i) the aggregate resource constraint is the relevant budget constraint, and (ii) the social planner internalizes the effect of borrowing on the interest rate.

The optimality conditions are:

$$u_c(c, l) = \lambda \quad (12a)$$

$$u_l(c, l) = -\lambda f_l(l) \quad (12b)$$

$$\beta - \frac{\dot{\lambda}}{\lambda} = i(z) + i_z(z)z \quad (12c)$$

$$\lim_{t \rightarrow \infty} \lambda z e^{-\beta t} = 0 \quad (12d)$$

The core dynamics are then modified to

$$\dot{z} = c + i(z)z - f(l) \quad (13a)$$

$$\dot{\lambda} = \lambda[\beta - \{i(z) + i_z(z)z\}] \quad (13b)$$

The corresponding steady-state conditions are:

$$f[l(\tilde{\lambda})] = c(\tilde{\lambda}) + i(\tilde{z})\tilde{z} \quad (14a)$$

$$\beta = i(\tilde{z}) + i_z(\tilde{z})\tilde{z} \quad (14b)$$

The critical difference between the outcome in the socially planned economy and the decentralized economy lies in (13b), or in the computation of the cost of additional borrowing from international capital markets. In the decentralized model, the private marginal cost of borrowing is zero since the agent treats $i(z)$ as exogenous in making its allocation decisions. By contrast, a social planner internalizes the positive marginal cost of borrowing ($i_z > 0$). Therefore, in the socially planned economy, the (social) cost of borrowing is the interest paid on the existing stock of debt, i , plus the marginal cost of borrowing an additional dollar, which in turn raises the cost of financing the entire stock of debt (the term $i_z z$). Since this marginal cost is not internalized by the private agent in the decentralized model, it generates an externality. This leads the decentralized economy to accumulate a higher stock of debt relative to the socially optimal level (given in 14b).

iv. *The optimal tax on foreign borrowing:*

Compare the steady state conditions (6b) in the decentralized economy and (14b) in the socially planned economy:

$$(1 + \tau)i(\tilde{z}) = i(\tilde{z}) + i_z(\tilde{z})\tilde{z} \quad (15)$$

Equation (15) can be solved for the optimal tax on foreign borrowing

$$\hat{\tau} = \frac{i_z(\tilde{z})\tilde{z}}{i(\tilde{z})} > 0 \quad (16)$$

As is evident from (16), the objective of levying a tax on borrowing in the decentralized economy is to equalize the private and social marginal cost of borrowing from international capital markets. Since the decentralized economy accumulates "too much" debt relative to the social optimum (because the private marginal cost is lower than the social cost), the optimal tax rate must be positive and, as seen in (16), equals the interest elasticity of foreign debt. Note that when international capital markets are perfect, $i_z = 0$ and consequently, $\hat{\tau} = 0$.