ECON 8510
SOLUTION / ASSIGNMENT 4

The Hamiltonian for the private agent is:

$$H = \frac{1}{\gamma}(C_T C_N^{\theta})^{\gamma} e^{-\rho t} + \lambda e^{-\rho t}\left[rb + Y_T - C_T + p\{Y_N - C_N - I\} - T_L - \dot{b}\right]$$

<u>Optimality conditions</u>: (Note that $k_i = K_i/L_i$; $i = N, T$)

$$C_T^{\gamma-1} C_N^{\theta\gamma} = \lambda \tag{1a}$$

$$\theta C_T^{\gamma} C_N^{\theta\gamma-1} = \lambda p \tag{1b}$$

$$\rho - \frac{\dot{\lambda}}{\lambda} = r \tag{1c}$$

$$\alpha A_T k_T^{\alpha-1} K_g^{\eta(1-\alpha)} = p \cdot \beta A_N k_N^{\beta-1} K_g^{\eta(1-\beta)} \tag{1d}$$

$$(1-\alpha) A_T k_T^{\alpha} K_g^{\eta(1-\alpha)} = p(1-\beta) A_N k_N^{\beta} K_g^{\eta(1-\beta)} \tag{1e}$$

$$\beta A_N k_N^{\beta-1} K_g^{\eta(1-\beta)} + \frac{\dot{p}}{p} = r \tag{1f}$$

[Note: (1f) is derived noting that $\dot{K} = I$ and $\partial H/\partial K_T = \partial H/\partial K_N = \partial H/\partial K$]
Finally, the usual transversality conditions apply for b and K.

Eq. (1a) and (1b) equate the MU of consumption from the traded and non-traded goods to the MU of wealth. From (1c), given exogenous ρ and r (world interest rate), we require that $\dot{\lambda} = 0$ $\forall t$, which implies a constant MU of wealth, i.e. $\lambda = \bar{\lambda}$, $\forall t$. Eqs. (1d) and (1e) equate the marginal returns from allocating capital and labor across the two sectors. Eq. (1f) represents a no-arbitrage condition for capital (non-traded). Since it is only non-traded capital that can be converted to investment, the total return to capital consists of the marginal product of non-traded capital plus ~~the~~ any capital gains/losses from fluctuations in its relative price, p.

From (1a) and (1b) we get

$$C_N = \frac{\theta}{p} C_T$$

Using the above in (1a) we can solve for C_T:

$$C_T = \left[\bar{\lambda}\left(\frac{p}{\theta}\right)^{\theta\gamma}\right]^{1/\gamma(1+\theta)-1} \qquad (2a)$$

Similarly,

$$C_N = \left[\bar{\lambda}\left(\frac{p}{\theta}\right)^{1-\gamma}\right]^{1/\gamma(1+\theta)-1} \qquad (2b)$$

Dividing (1d) by (1e) we get

$$k_N = \left[\frac{1-\alpha}{1-\beta}\cdot\frac{\beta}{\alpha}\right] k_T \qquad (3)$$

Using (3) in (1d), we can solve for k_T:

$$k_T = \Delta_1\, p^{1/\alpha-\beta} \cdot K_g^{\eta} \equiv k_T(p, K_g) \qquad (4a)$$

where

$$\Delta_1 = \left[\left(\frac{1-\alpha}{1-\beta}\cdot\frac{\beta}{\alpha}\right)^{\beta-1}\left(\frac{\beta \cdot A_N}{\alpha A_T}\right)\right]^{1/\alpha-\beta}$$

It immediately follows, from (3) that

$$k_N = \Delta_2\, p^{1/\alpha-\beta} K_g^{\eta} \equiv k_N(p, K_g) \qquad (4b)$$

where

$$\Delta_2 = \left[\left(\frac{1-\alpha}{1-\beta}\cdot\frac{\beta}{\alpha}\right)^{\alpha-1}\left(\frac{\beta A_N}{\alpha A_T}\right)\right]^{1/\alpha-\beta}.$$

Note that

$$K = K_T + K_N \qquad (5)$$

$$\Rightarrow \quad K = k_T L_T + k_N L_N \qquad (5')$$

Since $L_T + L_N = 1$, we can write (5') as

$$K = k_T L_T + k_N(1 - L_T) \qquad (5'').$$

Solving (5") for L_T:

$$L_T = \frac{K - k_N}{k_T - k_N}$$

Using (4a) and (4b),

$$L_T \equiv L_T(p, K, K_g) = \frac{K - k_N(p, K_g)}{k_T(p, K_g) - k_N(p, K_g)} \tag{6}$$

Eqns. (4a), (4b) and (6) enable us to characterize Y_N and Y_T:

$$Y_N = A_N k_N^{\beta} K_g^{\eta(1-\beta)}(1 - L_T) \equiv Y_N(p, K, K_g) \tag{7a}$$

$$Y_T = A_T k_T^{\alpha} K_g^{\eta(1-\alpha)} L_T \equiv Y_T(p, K, K_g) \tag{7b}$$

<u>Equilibrium Dynamics</u>:

There are 4 dynamic equations that describe the evolution of the economy. First, we have the goods market equilibrium condition in the non-traded sector:

$$Y_N = C_N + I = C_N + \mathring{K}$$

$$\Rightarrow \quad \mathring{K} = Y_N(p, K, K_g) - C_N(\bar{\lambda}, p) \tag{8a}$$

where $Y_N(\cdot)$ and $C_N(\cdot)$ are given by (7a) and (2b), respectively.

Second, we have the no-arbitrage condition for capital (non-traded) from (1f):

$$\mathring{p} = p\left[r - p A_N k_N^{\beta-1} K_g^{\eta(1-\beta)}\right] \tag{8b}$$

Third, the evolution of public capital is given by:

$$\mathring{K}_g = g\left[Y_T + pY_N\right] - \delta_g K_g \tag{8c}$$

Finally, the current account dynamics are given by

$$\mathring{b} = Y_T - C_T + rb \tag{8d}$$

Of the 4 equations (8a) - (8d), only 3 are independent ($\dot{K}, \dot{p}, \dot{K}_g$) while the evolution of the current account depends on the solution to (8a) - (8c). Therefore, the core dynamics are characterized by:

$$\dot{K} = Y_N(p, K, K_g) - C_N(\bar{\lambda}, p) \tag{9a}$$

$$\dot{K}_g = g\left[Y_T(p, K, K_g) + p \cdot Y_N(p, K, K_g)\right] - \delta_g K_g \tag{9b}$$

$$\dot{p} = p\left[r - \beta A_N k_N^{\beta-1} K_g^{\eta(1-\beta)}\right] \tag{9c}$$

The linearized dynamic system can be represented by

$$[\dot{\underline{X}}] = [J][\underline{X} - \tilde{\underline{X}}] \tag{10}$$

where $\underline{X}' = (K, K_g, p)$, $\tilde{\underline{X}}' = (\tilde{K}, \tilde{K}_g, \tilde{p})$ and $[J]$ is a 3×3 matrix of coefficients a_{ij} ($i,j = 1,2,3$). It can be demonstrated that the equilibrium is a saddle path, with K and K_g being the sluggish adjustment variables and p being the "jump" variable. The system will have 2 stable (negative) and 1 unstable (positive) eigenvalue.

Steady-State

The steady-state is attained when $\dot{K} = \dot{K}_g = \dot{p} = 0$ so that $K = \tilde{K}$, $K_g = \tilde{K}_g$ and $p = \tilde{p}$:

$$A_N \tilde{k}_N^{\beta} \tilde{K}_g^{\eta(1-\beta)}\left[1 - L_T(\tilde{p}, \tilde{K}, \tilde{K}_g)\right] = \left[\bar{\lambda}(\tilde{p}/\theta)^{1-\gamma}\right]^{1/\gamma(1+\theta)-1} \tag{11a}$$

$$g\left[A_T \tilde{k}_T^{\alpha} \tilde{K}_g^{\eta(1-\alpha)} L_T(\tilde{p}, \tilde{K}, \tilde{K}_g) + \tilde{p} A_N \tilde{k}_N^{\beta} \tilde{K}_g^{\eta(1-\beta)}\left\{1 - L_T(\tilde{p}, \tilde{K}, \tilde{K}_g)\right\}\right] = \delta_g \tilde{K}_g \tag{11b}$$

$$r = \beta A_N \tilde{k}_N^{\beta-1} \tilde{K}_g^{\eta(1-\beta)} \tag{11c}$$

Also, the following applies in the steady state:

$$\tilde{K} = L_T(\tilde{p}, \tilde{K}, \tilde{K}_g)\tilde{k}_T + \left[1 - L_T(\tilde{p}, \tilde{K}, \tilde{K}_g)\right]\tilde{k}_N \tag{11d}$$

$$(\dot{b} = 0) \quad A_T \tilde{k}_T^{\alpha} \tilde{K}_g^{\eta(1-\alpha)} L_T(\tilde{p}, \tilde{K}, \tilde{K}_g) - \left[\bar{\lambda}(\tilde{p}/\theta)^{\theta\gamma}\right]^{1/\gamma(1+\theta)-1} + r\tilde{b} = 0 \tag{11e}$$

~~We have 6 eqns which can be solved for the [illegible]:~~

Given that $k_T = k_T(p, K_g)$ and $k_N = k_N(p, K_g)$, eqns. (11a) – (11e) can be solved for $(\tilde{K}, \tilde{K}_g, \tilde{b}, \tilde{p}, \bar{\lambda})$. Once these are known, the sectoral allocations, $(\tilde{K}_T, \tilde{K}_N)$ and $(\tilde{L}_T, \tilde{L}_N)$ are immediately known.

Getting back to (9c):

$$\mathring{p} = p\left[r - \beta A_N k_N^{\beta-1} K_g^{\eta(1-\beta)}\right]$$

~~whereas~~ From (46) we see that

$$k_N = \Delta_2 \, p^{1/\alpha-\beta} K_g^{\eta}$$

$$\Rightarrow \quad \mathring{p} = p\left[r - \beta A_N \left\{\Delta_2 p^{1/\alpha-\beta} K_g^{\eta}\right\}^{\beta-1} K_g^{\eta(1-\beta)}\right]$$

$$\Rightarrow \quad \mathring{p} = p\left[r - \beta A_N \left\{\Delta_2 p^{1/\alpha-\beta}\right\}^{\beta-1} K_g^{\eta(\beta-1)} K_g^{\eta(1-\beta)}\right]$$

$$\Rightarrow \quad \boxed{\mathring{p} = p\left[r - \beta A_N \left\{\Delta_2 p^{1/\alpha-\beta}\right\}^{\beta-1}\right]} \qquad (12)$$

It is clear from (12) that $\mathring{p}$ is independent of K_g. Consequently, $\tilde{p}$ will also be independent of K_g. Therefore, government spending shocks, inspite of their productive nature, do not affect the real exchange rate (in transition or in the long-run).

The steady-state real exchange rate can be obtained from (12) by setting $\mathring{p} = 0$:

$$\tilde{p} = \left[\frac{\beta A_N}{r \, \Delta_2^{1-\beta}}\right]^{\alpha-\beta/1-\beta}$$

~~The~~ The real exchange rate is determined by supply-side variables only. Therefore, these results suggest that the standard two-sector dependent economy model's predictions are robust to modifications about the nature of demand shocks.