ECON 8010 PROBLEM SET #3 FALL 2009

SOLUTIONS

1. Using each of the utility functions below,

a. $U(X,Y) = X^{\frac{1}{3}}Y^{\frac{2}{3}}$ $X,Y > 0$

b. $U(X,Y) = -X^{-1} - Y^{-1}$ $X,Y > 0$

c. $U(X,Y) = \left(\log_e X\right) + Y$ $X > 1;\ Y > 0$

answer the following question:

1. Find the function, if it exists, for the marginal rate of substitution for each utility function.

a. $U(X,Y) = X^{\frac{1}{3}}Y^{\frac{2}{3}}$

$$MRS_{YX} = \frac{U_X}{U_Y} = \frac{\frac{1}{3}X^{-2/3}Y^{2/3}}{\frac{2}{3}X^{1/3}Y^{-1/3}} = \frac{Y}{2X}$$

b. $U(X,Y) = -X^{-1} - Y^{-1}$

$$MRS_{YX} = \frac{U_X}{U_Y} = \frac{X^{-2}}{Y^{-2}} = \frac{Y^2}{X^2}$$

c. $U(X,Y) = \left(\log_e X\right) + Y$

$$MRS_{YX} = \frac{U_X}{U_Y} = \frac{\frac{1}{X}}{1} = \frac{1}{X}$$

2. Are any of the above functions monotonic transformations of one another? Explain briefly.

   None of the above utility functions is a monotonic transformation of another. Since the MRS determines the shape of the indifference curve, the different MRS's of the utility functions show that they have distinct indifference curves and, thus, they are not equivalent in an ordinal sense.

3. Find a function for the marginal utility of income for each utility function. Is marginal utility increasing or decreasing in M for each utility function?

   a. $U(X,Y) = X^{\frac{1}{3}}Y^{\frac{2}{3}}$

$$\max_{X,Y} U(X,Y) = X^{\frac{1}{3}}Y^{\frac{2}{3}} \ s.t. \ X \cdot P_X + Y \cdot P_Y = M$$

$$L = X^{\frac{1}{3}}Y^{\frac{2}{3}} + \lambda(M - X \cdot P_X + Y \cdot P_Y)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= \tfrac{1}{3} X^{-\frac{2}{3}}Y^{\frac{2}{3}} - \lambda P_X = 0 \\ \frac{\partial L}{\partial y} &= \tfrac{2}{3} X^{\frac{1}{3}}Y^{-\frac{1}{3}} - \lambda P_Y = 0 \end{aligned}\right\} MRS_{YX} = \frac{\frac{1}{3} X^{-\frac{2}{3}}Y^{\frac{2}{3}}}{\frac{2}{3} X^{\frac{1}{3}}Y^{-\frac{1}{3}}} = \frac{Y}{2X} = \frac{P_X}{P_Y} \ \rightarrow \ Y = 2\frac{P_X}{P_Y}X$$

$$\frac{\partial L}{\partial \lambda} = M - X \cdot P_X + Y \cdot P_Y = 0 \ \rightarrow \ M = X \cdot P_X + Y \cdot P_Y$$

$$\rightarrow \ M = X \cdot P_X + 2\frac{P_X}{P_Y} X \cdot P_Y = 3P_X X$$

$$\rightarrow \ X^* = \frac{M}{3P_X}, \quad Y^* = 2X^* \frac{P_x}{P_y} = \frac{2M}{3P_Y}$$

From the FOC's, $\lambda^* = \dfrac{\frac{1}{3} X^{*-\frac{2}{3}} Y^{*\frac{2}{3}}}{P_X} = \dfrac{\frac{2}{3} X^{\frac{1}{3}} Y^{-\frac{1}{3}}}{P_Y} = \dfrac{\frac{2}{3}\left(\frac{M}{3P_X}\right)^{\frac{1}{3}}\left(\frac{2M}{3P_Y}\right)^{-\frac{1}{3}}}{P_Y} = \dfrac{2^{2/3}}{3P_X^{1/3}P_Y^{2/3}}$

Therefore, $\dfrac{\partial \lambda^*}{\partial M} = 0$. Hence, the marginal utility of income is independent of M.

b. $U(X,Y) = -X^{-1} - Y^{-1}$

$$\max_{X,Y} U(X,Y) = -X^{-1} - Y^{-1} \; s.t. \; X \cdot P_X + Y \cdot P_Y = M$$

$$L = -X^{-1} - Y^{-1} + \lambda(M - X \cdot P_X + Y \cdot P_Y)$$

$$\left.\begin{array}{l} \frac{\partial L}{\partial x} = X^{-2} - \lambda P_X = 0 \\ \frac{\partial L}{\partial y} = Y^{-2} - \lambda P_Y = 0 \end{array}\right\} MRS_{YX} = \frac{X^{-2}}{Y^{-2}} = \frac{Y^2}{X^2} = \frac{P_X}{P_Y} \rightarrow Y = \sqrt{\frac{P_X}{P_Y}} X$$

$$\frac{\partial L}{\partial \lambda} = M - X \cdot P_X + Y \cdot P_Y = 0 \rightarrow M = X \cdot P_X + Y \cdot P_Y$$

$$\rightarrow M = X \cdot P_X + \frac{P_X^{1/2}}{P_Y^{1/2}} X \cdot P_Y = X\left(P_X + P_X^{1/2} P_Y^{1/2}\right) = X \cdot P_X^{1/2}\left(P_X^{1/2} + P_Y^{1/2}\right)$$

$$\rightarrow X^* = \frac{M}{P_X^{1/2}\left(P_X^{1/2} + P_Y^{1/2}\right)}, \; Y^* = \sqrt{\frac{P_X}{P_Y}} \cdot X^* = \frac{M}{P_Y^{1/2}\left(P_X^{1/2} + P_Y^{1/2}\right)}$$

From the FOC's, $\lambda^* = \frac{1}{X^{*2} P_X} = \frac{1}{Y^{*2} P_Y} = \frac{1}{P_Y \left(\frac{M}{P_Y^{1/2}\left(P_X^{1/2} + P_X^{1/2}\right)}\right)^2} = \frac{\left(P_X^{1/2} + P_Y^{1/2}\right)^2}{M^2}$

Therefore, $\frac{\partial \lambda^*}{\partial M} = -2 \frac{\left(P_X^{1/2} + P_Y^{1/2}\right)^2}{M^3} < 0$. Hence, the marginal utility of income is decreasing in M.

c. $U(X,Y) = \left(\log_e X\right) + Y$

$$\max_{X,Y} U(X,Y) = \left(\log_e X\right) + Y \ \ s.t.\ X \cdot P_X + Y \cdot P_Y = M$$

$$L = \left(\log_e X\right) + Y + \lambda(M - X \cdot P_X + Y \cdot P_Y)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= \frac{1}{X} - \lambda P_X = 0 \\ \frac{\partial L}{\partial y} &= 1 - \lambda P_Y = 0 \end{aligned}\right\} MRS_{YX} = \frac{\frac{1}{X}}{1} = \frac{1}{X} = \frac{P_X}{P_Y} \rightarrow X = \frac{P_Y}{P_X}$$

$$\frac{\partial L}{\partial \lambda} = M - X \cdot P_X + Y \cdot P_Y = 0 \rightarrow M = X \cdot P_X + Y \cdot P_Y$$

$$\rightarrow M = \frac{P_Y}{P_X} \cdot P_X + Y \cdot P_Y = P_Y\left(1 + Y\right)$$

$$\rightarrow Y^* = \frac{M}{P_Y} - 1, \ \ X^* = \frac{P_Y}{P_X}$$

From the FOC's, $\lambda^* = \frac{1}{X}\frac{1}{P_X} = \frac{1}{P_Y}$. Therefore, $\frac{\partial \lambda^*}{\partial M} = 0$. Hence, the marginal utility of income is independent of M.