Developments in Multi-Attribute Portfolio Selection

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Abstract

In this paper we reconcile why it is possible that people in finance view conventional portfolio selection as a single criterion problem and people in multiple criteria optimization view it as a bi-criterion problem. We then show how, for more complex investors, the theory of mean-variance portfolio selection can be extended to include additional objectives such as dividends, liquidity, turnover, number of securities in a portfolio, and so forth. This is followed by a discussion of the nature of the nondominated sets of multiple objective portfolio selection problems and developments underway for the solution of such problems.

Keywords: Portfolio selection, efficient frontiers, multiple objectives, equivalent deterministic problems, nondominated sets, ε-constraint methods, hyperboloidic platelets

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1 Introduction

It is a rare person in finance that sees portfolio selection as a multiple criteria problem. And it is a rare person in multiple criteria optimization that sees portfolio selection as a single criterion problem.

When finance people think of portfolio selection, they typically think of it in terms of the "mean-variance" formulation

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} \,\} \\ &\text{s.t.} \quad \boldsymbol{\mu}^T \mathbf{x} \geq \rho \\ &\qquad \mathbf{x} \in S \end{aligned} \tag{1}$$

in which ρ is to be parameterized over a wide enough range to compute the "efficient frontier" (but what we will call the "nondominated frontier"). In practice, most people in finance settle for the repetitive solving of (1) for different values of ρ to obtain a dotted or piecewise linear characterization of the nondominated frontier. Twenty such optimizations would not be uncommon. See Figure 1. However, when (1) is viewed by a multiple criteria optimization person, it is recognized as an $\boldsymbol{\varepsilon}$-constraint program. An $\boldsymbol{\varepsilon}$-constraint program is a multiple objective program that has been reformulated for solution as a single objective problem in which all but one of the objectives have been converted to constraints (see for example Steuer [20], Chap. 8). Thinking of the problem behind the $\boldsymbol{\varepsilon}$-constraint program, a multiple criteria optimization person sees the "mean-variance" problem of (1) more aptly expressed as

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} \,\} \\ &\max\{\, \boldsymbol{\mu}^T \mathbf{x} \} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{2}$$

in which the endeavor is to compute all points in S that are *efficient* so as to obtain, by taking their images, the *nondominated set* (or in this case because there are only two objectives, the "nondominated frontier"). Either way, with the same intended solution sets, (1) and (2) are simply two different ways of expressing the same thing.

To make clear the notation employed above along with the basic problem of portfolio selection attributable to Markowitz [13, 14], let

(a) there be a beginning of a holding period

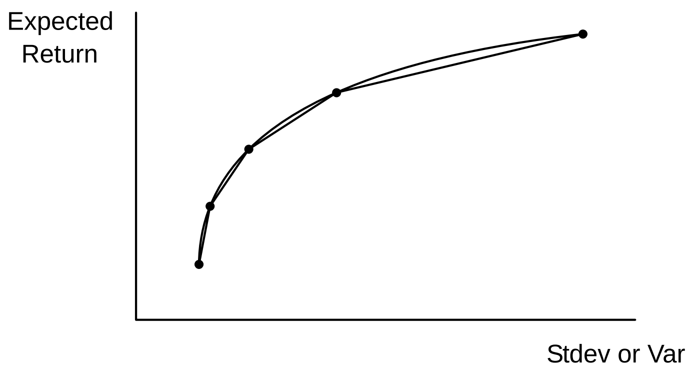

Figure 1: Nondominated frontier (curved line) along with, using only five points, dotted and piecewise linear representations. Note that oftentimes nondominated frontiers are portrayed with standard deviation rather than variance on the horizontal axis. The vertical axis is always expected return.

(b) there be an end of the holding period
(c) there be an initial sum to be invested
(d) n be the number of securities in the pool from which a portfolio is to be formed
(e) $\mathbf{x} = (x_1, \ldots, x_n)$ denote a *portfolio* where x_i specifies the proportion of the sum invested in security i
(f) $S \subset \mathbb{R}^n$ be the set of all feasible portfolios which often is expressed, as assumed in this paper, as simply as $S = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1,\ 0 \leq x_i \leq 1\}$.

With expected value $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $n \times n$ covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & & \\ \vdots & & & \vdots \\ \sigma_{n1} & & \cdots & \sigma_{nn} \end{bmatrix},$$

let $\mathbf{r} = (r_1, \ldots, r_n)$ be the random vector that specifies the returns of the n securities to be realized over the course of the holding period. While the realized values of the r_i are not known until the end of the holding period, basic theory assumes that $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$, also known as "Markowitz inputs," are known with certainty at the beginning of the holding period.

The rest of the paper is organized as follows. In Section 2, we show how both finance and multiple criteria optimization can each be correct in viewing portfolio selection with one and two "objectives," respectively. In Section 3 we discuss the introduction of multiple objectives into the theory of portfolio selection, and in Section 4 we demonstrate the platelet-wise hyperboloidic nature of the nondominated set in multiple objective portfolio selection programming. Section 5 comments on a Java code under development and Section 6 ends the paper with concluding remarks.

2 Two Levels of Assumptions

We note that neither (1) nor (2) is a starting point. Rather, the formulations, which represent two ways of writing the same problem, are a consequence of two levels of assumptions.

In conventional theory, the assumptions at the highest level are that the self-interest model of economics applies and that markets are efficient. This means that investors need only concern themselves with "making money" (the more the better). The belief here is that there is no need to take into account factors such as dividends, quality of corporate governance, social responsibility, and so forth, as all such effects are assumed to be already in the prices.

Portraying the situation of conventional portfolio selection, we have Figure 2. At the top is the investor's *overall focus*, to make money. Commencing the operationalization of the overall focus then results in the formulation

$$\begin{aligned} &\max\{\, \mathbf{r}^T\mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{3}$$

in which the objective of *portfolio return*, given by $\mathbf{r}^T\mathbf{x}$, is noted to be a random variable. This is because $\mathbf{r}^T\mathbf{x}$ involves the random vector $\mathbf{r}$. Consequently (3), which might look like a linear program, is not a linear program. It is a *stochastic program* as a result of its objective being stochastic. Thus we see the difficulty. While the $\mathbf{r}$ are not known until the end of the holding period, $\mathbf{x} \in \mathbb{R}^n$ must nevertheless be selected at the beginning of the holding period.

Since a stochastic program cannot be solved in its present form, the assumptions at the second level involve how to replace (3) with an *equivalent deterministic program.* Caballero, Cerdá, Muñoz, Rey and Stancu-Minasian

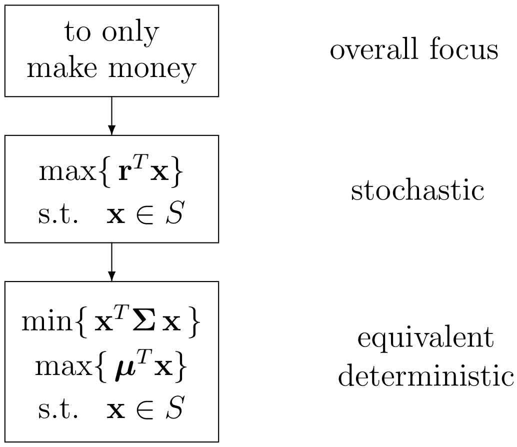

Figure 2: Hierarchical structure of the overall focus, stochastic, and equivalent deterministic stages in conventional portfolio selection.

[3], enumerate several possibilities, the third of which is mean-variance. Thus, and bear in mind that this was before either stochastic programming or multiple criteria optimization appeared on anybody's radar screen, the contributions of Markowitz's were (a) his choice of the problem of mean-variance (regardless of whether it is in the form of (1) or (2)) as the equivalent deterministic problem and (b) his protocol of computing, and then searching for the most preferred point on, the nondominated frontier.

We are now able to reconcile the two viewpoints and the two formulations. Looking at portfolio selection from an overall focus perspective, finance only sees the single stochastic objective of portfolio return. Note that writing the equivalent deterministic problem in the form of (1) only tends to reinforce the single criterion viewpoint. On the other hand, multiple criteria optimization, taking the equivalent deterministic problem in the form of (2) at face value, sees portfolio selection as possessing the two deterministic objectives of expected return and variance. How many objectives ones sees depends upon whether one is looking at the equivalent deterministic program or not.

3 Multiple Criteria Portfolio Theory

With the overall focus, stochastic, and equivalent deterministic stages of portfolio selection as developed in the previous section, we now show where and how multiple objectives can enter the picture. Multiple criteria portfolio selection is for at least three groups of investors. One group would consist of investors who believe in the self-interest model and efficient markets, but do not believe in the 100% certainty assumption about the Markowitz inputs at the beginning of the holding period. Such investors might well wish to monitor their portfolios with regard to other measures such as dividends, growth in sales, amount invested in R&D, and so forth, in order to hedge against errors that might be made by selecting portfolios based upon expected return and variance alone. Another group would consist of people who do not believe, or only partially believe, in efficient markets. For them, prices do not always reflect all known information as they see it and they might wish to identify value and desirability using additional or other measures. A third group would consist of investors also interested in "portfolio-as-a-whole" criteria as pointed out in Polyashuk [18]. Consequently, consider the two groups of possible objectives

$$
\begin{aligned}
&\max\{\, z_1 = \text{portfolio return}\} \\
&\max\{\, z_2 = \text{dividends}\} \\
&\max\{\, z_3 = \text{amount invested in R\&D}\} \\
&\max\{\, z_4 = \text{social responsibility}\} \\
&\max\{\, z_5 = \text{liquidity}\}
\end{aligned}
$$

$$
\begin{aligned}
&\min\{\, z_6 = \text{deviations from target asset allocation percentages}\} \\
&\min\{\, z_7 = \text{number of securities in portfolio}\} \\
&\min\{\, z_8 = \text{turnover (i.e., costs of adjustment)}\} \\
&\min\{\, z_9 = \text{maximum investment proportion weight}\} \\
&\min\{\, z_{10} = \text{amount of short selling}\}
\end{aligned}
$$

In the first group, the criteria are derived from random variable attributes of the individual securities, and are thus stochastic. In the second group, the criteria are derived from properties of the portfolio as a whole. That is, the values of the criteria can be ascertained with certainty upon inspection of the $\mathbf{x}$-vector under question, and are thus deterministic.

The overall focus, stochastic, and equivalent deterministic stages of portfolio selection with multiple criteria are shown in Figure 3. To illustrate, assume that an investor has in his or her overall focus portfolio return and dividends as objectives. Let portfolio return be split into mean and variance as usual in accordance with the third choice enumerated by Caballero, et al. However, since the variability of dividends is of much less importance than the variability of portfolio return, let us assume that dividends can be adequately represented by its mean vector in the equivalent deterministic program in accordance with the first choice enumerated by Caballero et al. In this multiple objective illustration, the resulting equivalent deterministic program would then be of a 1-quadratic-2-linear variety.

A few more comments before leaving Figure 3. The first set of three vertical dots in the second box of Figure 3 refers to other stochastic objectives that are to be split into mean and variance pairs in the third box. Along with $\max\{\mathbf{c}^T\mathbf{x}\}$, the second set of three vertical dots in the second box refer to either (a) stochastic objectives that can be replaced in the third box by a single deterministic objective (such as mean) or (b) originally deterministic objectives (such as the portfolio-as-a-whole objectives in the above list) that merely need to be re-copied into the third box as a result of the these objectives being deterministic in the first place.

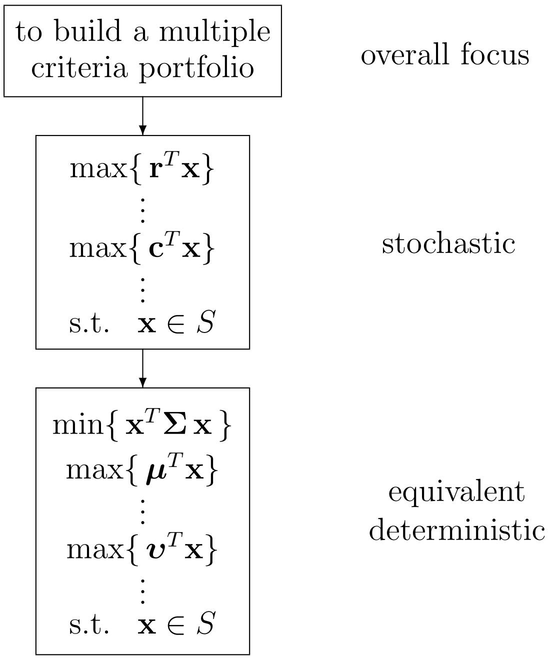

Figure 3: Hierarchical structure of the overall focus, stochastic, and equivalent deterministic stages in multiple criteria portfolio selection where the expected value vector of $\mathbf{r}$ is $\boldsymbol{\mu}$ and the expected value vector of $\mathbf{c}$ is $\boldsymbol{\upsilon}$.

4 Piecewise Hyperbolic and Hyperboloidic

Before commenting on methods under development for solving equivalent deterministic programs with one quadratic and two or more linear objective functions, let us say a few things about the 1-quadratic-1-linear equivalent deterministic problems of conventional portfolio selection. As depicted in Figure 4, the efficient set is piecewise linear in **x**-space as on the left, and the nondominated set is piecewise *hyperbolic* in (standard deviation, expected return) space[1] as on the right .

As for equivalent deterministic problems with one quadratic and two or more linear objective functions that can easily arise in multiple criteria portfolio selection, the efficient set is a connected union of polyhedral sets in **x**-space and the nondominated set is platelet-wise *hyperboloidic* in (standard deviation, expected return, extra linear objective) space[2]. This is as in Figure 5 with the efficient set portrayed on the left and the platelet-wise (like on the back of a turtle) nondominated set portrayed on the right.

Whereas methods for solving for the nondominated frontiers of mean-variance problems have been well studied, methods for solving for the nondominated surfaces of equivalent deterministic problems with additional linear objective functions are only under development (as being worked on for instance by Fliege [5], Kliber [10] and Hirschberger, Qi and Steuer [7, 8]).

5 Java Code

One of the items under development in [7, 8] is a Java code for portfolio selection. The purposes of the code are that it

- **(a)** be fast and easy to use
- **(b)** be able to address large-scale conventional and multiple objective portfolio selection problems
- **(c)** be able to compute all hyperbolic segments or hyperboloidic platelets of the nondominated set
- **(d)** be able to handle covariance matrices that are up to 100% dense in nonzero elements

[1]or piecewise *parabolic* in (variance, expected return) space

[2]or platelet-wise *paraboloidic* in (variance, expected return, extra linear objective) space

(e) be equipped with a built-in random portfolio-selection problem generator.

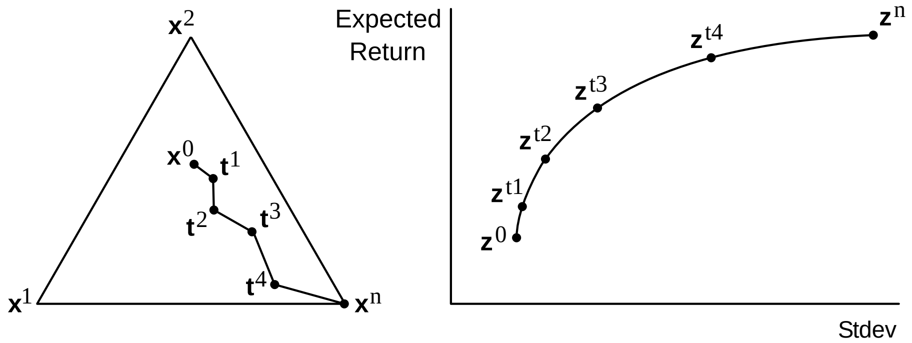

Figure 4: Portrayal of the efficient and nondominated sets of a conventional mean-variance portfolio selection problem.

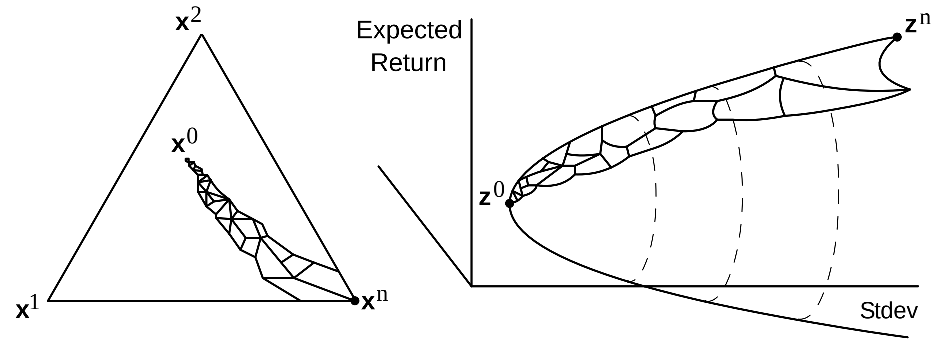

Figure 5: Portrayal of the efficient and nondominated sets of an equivalent deterministic problem with one quadratic and two linear objective functions.

This means that with this code it is no longer necessary in large problems to diagonalize the covariance structure, and endure the resulting inevitable loss of information, to achieve computational feasibility as the code can handle 100% dense covariance matrices directly. Moreover, because the code provides for the exact computation of the nondominated set, it is no longer necessary to utilize ε-constraint methods to obtain approximations of the nondominated set as are our only choices when using software such as Matlab [16], Mathematica [23], Cplex [4], LINGO [19], or SAS. Unfortunately, a code for the excellent procedure described in Markowitz, Todd, Xu and Yamane [15] is not known to be available.

Although the code is not yet available for distribution, some preliminary computational results can be reported. For instance, in a normal (i.e., 1-quadratic-1-linear) mean-variance portfolio optimization problem

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x}\, \} \\ &\max\{\, \boldsymbol{\mu}^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned}$$

with $n = 500$, $S = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ 0 \leq x_i \leq 1\}$, and a 100% dense $\boldsymbol{\Sigma}$, one would expect about 200 nondominated hyperbolic segments. But in a 1-quadratic-2-linear problem

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x}\, \} \\ &\max\{\, \boldsymbol{\mu}^T \mathbf{x}\} \\ &\max\{\, \boldsymbol{\upsilon}^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned}$$

with the same n, S and $\boldsymbol{\Sigma}$, one would expect about 5,000 nondominated hyperboloidic platelets.

With regard to the last item in the above list, the Java code under development contains a built-in random problem generator. Using the generator, portfolio selection problems with one quadratic and one or more linear objective functions can be randomly generated for any number of securities up to at least 3,000. What is non-trivial about the random problem generation task is how to generate the $\boldsymbol{\Sigma}$ covariance matrices. Unfortunately, it is nearly impossible to create covariance matrices larger than about 20×20 by simply assigning random numbers. The reason is that for a matrix to be a covariance matrix, it must be positive semidefinite. To save a user from having to resort to different universes of historical data to obtain valid $\boldsymbol{\Sigma}$s, the built-in random problem generator employs a method for randomly generating realistic covariance matrices that have pre-chosen distributional characteristics. The method employed is described in Hirschberger, Qi and Steuer [9]. With this capability, the code should be highly useful for computational and benchmark testing purposes in the portfolio optimization area.

6 Future Directions

As exemplified by the works of Aouni, Ben Abdelaziz and El-Fayedh [1], Bana e Costa and Soares [2], Hallerbach and Spronk [6], Lo, Petrov and Wierzbicki [12], Ogryczak [17], Steuer, Qi and Hirschberger [21], Xu and Li [24], and others, the general area of multi-attribute portfolio selection has begun to attract increased attention. With regard to the content of this paper, we see two particularly fertile areas for future research. One is on further work to compute the nondominated sets of multiple criteria portfolio selection problems. The other is on how to search a multiple criteria portfolio selection nondominated surface for a most preferred point on it.

With regard to computing or characterizing nondominated sets, what we have done in this paper is talk about the easy case, when there is one quadratic and all other objective functions are linear. More difficult cases would involve equivalent deterministic problems (a) in which there are several quadratic and several linear objective functions, or (b) in which one or more of the non-quadratic objective functions are discrete or non-smooth. In the first case, a weighted-sums objective function could probably be formed so as to facilitate the repetitive sampling of the nondominated set using the Java code. Admittedly, this could involve many optimizations and consume considerable CPU time. In the second case, evolutionary algorithms such as employed in Streichert, Ulmer and Zell [22] would presumably be necessary to obtain a discretized representation of the nondominated set.

Either way, the nondominated sets of most multiple criteria portfolio optimization problems are likely to be known only via a number of given points (ideally, a very very large number of given points). Then the task becomes how to search among perhaps tens or hundreds of thousands of points to find a most preferred. Among several methods that might be considered is the projected line search method proposed in Korhonen and Karaivanova [11].

As one can see from this paper, with multiple criteria portfolio selection only now emerging, much work remains yet to be done.

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