

# A Correction to the Connectedness of the Evans-Steuer Algorithm of Multiple Objective Linear Programming

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## Abstract

The Evans-Steuer algorithm (and its ADBASE computerized implementation) for finding all efficient extreme points of a multiple objective linear program (MOLP) is based on efficient pivots. The justification of the algorithm rests on the assumption that the set of efficient basic index sets is connected with respect of efficient pivots. This assumption is incorrect. Nevertheless, the algorithm is valid and is, in fact, more efficient than originally believed.

**Keywords:** Multiple objective linear programming, ADBASE, efficient pivots, efficient bases, efficient extreme points

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# 1 Introduction

In multiple criteria decision making there are algorithms for characterizing the efficient set of the multiple objective linear program (MOLP)

$$\begin{aligned} \text{“max” } \mathbf{C}\mathbf{x} & & (1) \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

by computing all efficient extreme points. Such algorithms include those by Evans and Steuer [5], Zeleny [21], Isermann [10], Gal [8], Ecker and Kouada [3], Yu [20], and Armand and Malivert [1]. Of the algorithms developed by these authors, only two have ever been successfully enough programmed to have been made available for distribution. One was EFFACET [14] which implemented Isermann’s algorithm. However, EFFACET was available for only a few years during the 1980s. The other is ADBASE [15] which implements the Evans and Steuer algorithm. ADBASE has been continuously available since 1974. With the ISI Web of Science listing over 100 articles in which ADBASE is cited, ADBASE has been the dominant computer code for computing all efficient extreme points since its inception. Underscoring the extent to which multiple objective programming has begun to permeate other disciplines is the range of applications observed over the 100 citations. For instance since 2003, in addition to citations from economics, management, and operations research [2, 7, 11, 17, 6], there are also citations from engineering [22], biochemistry [19], biotechnology [18], and radiation therapy [4]. Having undergone numerous revisions over the years, in its current version ADBASE is dimensioned for MOLPs with up to 500 constraints, 900 structural variables, 20 objectives, and 1 million efficient extreme points. With ADBASE frequently utilized as the lone computer code for computing all efficient extreme points, this paper concentrates on an inadvertent representation about the Evans-Steuer algorithm that needs to be “set straight” for posterity.

In the Evans-Steuer algorithm for finding all efficient extreme points of (1), the idea of an *efficient pivot* plays a central role. This is developed and defined as follows.

**Definition 1** *Assume (1). Let  $\mathbf{A}$  be of full row rank and  $\mathfrak{B}$  be a basic index set with corresponding canonical tableau as in Table 1 where  $\mathbf{x}_B$  is the basic*

and  $\mathbf{x}_N$  the non-basic vector. Then  $\mathfrak{B}$  is a feasible basic index set if and only if  $\mathbf{x}_B \geq \mathbf{0}$ .

	$\mathbf{x}_B$	$\mathbf{x}_N$	
	$\bar{\mathbf{I}}$	$\bar{\mathbf{D}}$	$\mathbf{x}_B$
$c_j - z_j$	$\mathbf{0}$	$\bar{\mathbf{C}}_N$	

Table 1: Canonical tableau

**Definition 2** Let  $\mathfrak{B}$  and  $\mathfrak{B}'$  be basic index sets. Then  $\mathfrak{B}$  and  $\mathfrak{B}'$  are adjacent if and only if they differ in one element.

**Definition 3** Let  $\mathfrak{B}$  be a feasible basic index set. Then  $\mathfrak{B}$  is efficient if for some  $\boldsymbol{\lambda} > \mathbf{0}$ ,

$$\boldsymbol{\lambda}^T [\mathbf{C}_N - \mathbf{C}_B \bar{\mathbf{D}}] \leq \mathbf{0}. \quad (2)$$

**Definition 4** Let  $\mathfrak{B}$  be an efficient basic index set and  $x_j$  be a non-basic variable. If there exists some  $\boldsymbol{\lambda} > \mathbf{0}$  satisfying, in addition to (2), the condition that  $\boldsymbol{\lambda}^T [\mathbf{C}_N - \mathbf{C}_B \bar{\mathbf{D}}]$  has component zero in the  $x_j$  column, then the operation of bringing  $j$  into the basic index set while maintaining feasibility is called an efficient pivot.

A pivot operation on the tableau of a feasible basic index set “maintains feasibility” if conducted in one of these two ways:

1. the pivot element is positive and the usual minimum ratio rule of linear programming is satisfied.
2. the pivot element is any non-zero number and the row is degenerate; i.e., the constant term in the corresponding equation is zero.

Evans and Steuer in [5], Steuer in [16], and Hartley in [9] all offer proofs of the assertion that the set of all efficient basic index sets is connected with respect to efficient pivots. We show that this statement is incorrect but that, in fact, the Evans-Steuer algorithm does work, but for a different reason.

## 2 Connected Bases

In this section we outline some known (and correct) results about connected basic index sets and outline their proofs in order to “set the record straight”.

Note that two feasible basic index sets are adjacent if and only if one may be obtained from the other by a pivot that maintains feasibility. The following is a well known result. It is stated, e.g., in Mañas and Nedoma [12].

**Theorem 1** *Let  $\mathfrak{B}$  and  $\bar{\mathfrak{B}}$  be feasible basic index sets for (1). Then there exist feasible basic index sets  $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_p$  such that  $\mathfrak{B}_1 = \mathfrak{B}$ ,  $\mathfrak{B}_p = \bar{\mathfrak{B}}$  and  $(\mathfrak{B}_k, \mathfrak{B}_{k+1})$  is an adjacent pair,  $k = 1, 2, \dots, p - 1$ .*

*Proof:* Since the result is known, only a brief outline of the proof is given. Let  $\mathbf{r}$  be the extreme point corresponding to  $\bar{\mathfrak{B}}$ . Then there exists a vector  $\mathbf{c}$  such that the problem

$$\begin{aligned} \max \mathbf{c}^T \mathbf{x} & & (3) \\ \mathbf{A}\mathbf{x} = \mathbf{b} & \\ \mathbf{x} \geq \mathbf{0} & \end{aligned}$$

has the unique solution  $\mathbf{r}$ , namely,  $c_i = 0$  if  $i \in \bar{\mathfrak{B}}$  and  $-1$  otherwise. Solve this problem by the simplex algorithm starting with feasible basic index set  $\mathfrak{B}$ . The simplex algorithm gives a chain of adjacent bases terminating with a basic index set  $\mathfrak{B}'$  corresponding to  $\mathbf{r}$ . If  $\mathbf{r}$  is non-degenerate then  $\mathfrak{B}' = \bar{\mathfrak{B}}$  and we are finished. If not, then it is not hard to show that there exists a chain of adjacent feasible basic index sets leading from  $\mathfrak{B}'$  to  $\bar{\mathfrak{B}}$ . (To get from one of these to the next, one must pivot in a degenerate row.)

Using this theorem, it is possible to devise a systematic method for enumerating all feasible basic index sets of (1). This is described in Mañas and Nedoma [12], too. This method can furthermore, as pointed out in Section 12.2 of Martos [13], be extended to find all unbounded edges. This extension is based on the following well known (and correct) characterization of the unbounded edges of the feasible region of (3):  $E$  is an unbounded edge if and only if there exists a feasible basic index set  $\mathfrak{B}$  and a non-basic variable  $x_k$  for which  $E$  can be described by the conditions  $x_j = 0$ , for all  $j$  not in  $\mathfrak{B}$ ,  $j \neq k$ , and  $x_k \geq 0$ . We recognize in a canonical tableau an unbounded edge by seeing in  $\bar{\mathbf{D}}$  a column  $\leq \mathbf{0}$ .

The method cited above for finding all basic index sets and all unbounded edges of the feasible region of (3) gives rise to a method for finding all optimal extreme points and all optimal unbounded edges of (3). Let  $\mathfrak{B}$  be an optimal basic index set. Then for all  $\mathbf{x}$  feasible for (3)

$$\mathbf{c}^T \mathbf{x} = \sum_{j \notin \mathfrak{B}} \bar{c}_j x_j + z^*$$

where  $\bar{c}_j \leq 0$  for all  $j$  not in  $\mathfrak{B}$  and  $z^*$  is the optimal value for (3). Let  $\mathfrak{E} = \{j \mid j \notin \mathfrak{B}, \bar{c}_j < 0\}$ . Then a feasible solution for (3) is optimal if and only if  $x_j = 0$  for all  $j$  in  $\mathfrak{E}$ . Consider the *reduced system* obtained by deleting from the constraints all columns corresponding to variables in  $\mathfrak{E}$ . The optimal extreme points of (3) are exactly the vectors obtained by taking the extreme points of the reduced system and putting zero values for the deleted variables. Similarly, by finding the unbounded edges of the reduced system, we can find the unbounded optimal edges of (3).

Note that, while finding all optimal extreme points of (3), this method does not always produce all optimal basic index sets of (3). For example, let Table 2 be an optimal tableau for (3). Here the method just described yields the last two columns as the reduced system with  $\{1, 2\}$  as the only optimal basic index set, while pivoting as indicated in the full system shows that  $\{1, 4\}$  is also an optimal basic index set.

	$\mathbf{x}_B$		$\mathbf{x}_N$		
	1	0	1	1	1
	0	1	1	-1	0
$c_j - z_j$	0	0	-2	-1	0

Table 2: A single-objective linear program whose optimal basic index sets are not connected by optimal pivots

Finally we note that the method we have described for finding all optimal extreme points and all optimal unbounded edges is identical to that in Chapter 4 of Steuer [16] except that in [16] it is claimed incorrectly that the method finds *all* optimal basic index sets.

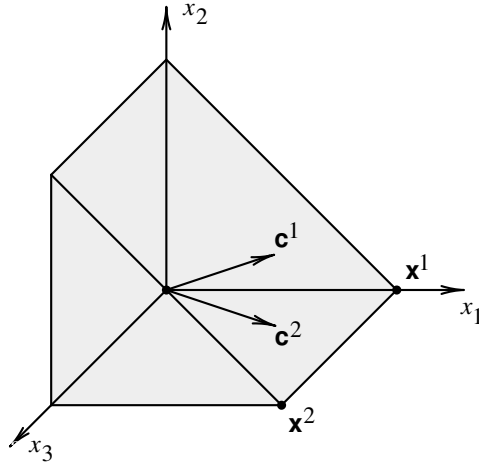


Figure 1: Multiple objective, multiple efficient extreme point illustration

### 3 A Multiple Objective Example

To show that the situation of Table 2 is not just a single-objective linear programming phenomenon, an MOLP example with multiple efficient extreme points is now provided. Consider

$$\begin{aligned}
 \max \{ 3x_1 + x_2 &= z_1 \} \\
 \max \{ 3x_1 - x_2 &= z_2 \} \\
 \text{s.t. } x_1 + x_2 &\leq 4 \\
 x_1 - x_2 &\leq 4 \\
 x_3 &\leq 4 \\
 x_1, x_2, x_3 &\geq 0
 \end{aligned}$$

whose graph is in Figure 1. This problem has two efficient extreme points  $\mathbf{x}^1 = (4, 0, 0)$  and  $\mathbf{x}^2 = (4, 0, 4)$ , and four efficient basic index sets  $\{1, 5, 6\}$ ,  $\{1, 3, 5\}$ ,  $\{1, 2, 6\}$ , and  $\{1, 2, 3\}$ . A set of tableaus for computing all efficient extreme points is in Table 3.

From the first tableau, we pivot as indicated to the second tableau. The second tableau yields our first efficient basic index set  $\{1, 5, 6\}$  and efficient extreme point  $\mathbf{x}^1$ . In this tableau there is only one possible efficient pivot as indicated. Pursuing the efficient pivot we obtain the third tableau which yields efficient basic index set  $\{1, 3, 5\}$  and efficient extreme point  $\mathbf{x}^2$ . In this

$x_1$	$x_2$	$x_3$	$s_4$	$s_5$	$s_6$		
$\boxed{1}$	1	0	1	0	0	4	$s_4$
1	-1	0	0	1	0	4	$s_5$
0	0	1	0	0	1	4	$s_6$
3	1	0	0	0	0		
3	-1	0	0	0	0		

1	1	0	1	0	0	4	$x_1$
0	-2	0	-1	1	0	0	$s_5$
0	0	$\boxed{1}$	0	0	1	4	$s_6$
0	-2	0	-3	0	0		
0	-4	0	-3	0	0		

1	1	0	1	0	0	4	$x_1$
0	-2	0	-1	1	0	0	$s_5$
0	0	1	0	0	$\boxed{1}$	4	$x_3$
0	-2	0	-3	0	0		
0	-4	0	-3	0	0		

Table 3: Multiple objective simplex tableaus that find all efficient extreme points

tableau there is only one possible efficient pivot, but the efficient pivot leads back to the efficient basic index set from which we have just come. With no other un-pursued efficient pivots remaining, we are done. Thus we see that it is possible to compute all efficient extreme points in a multiple objective situation without always visiting all efficient basic index sets.

## 4 Evans-Steuer Algorithm Revisited

In [1], Evans and Steuer present an algorithm (further described in Chapter 9 of [16]) for finding all efficient extreme points and all efficient unbounded edges of (1). The algorithm may be described as follows. We maintain three lists. The first list consists of processed efficient basic index sets, the second of unprocessed efficient basic index sets, and the third of unbounded edges. We start by finding an efficient basic index set and we put this on the processed list. All adjacent basic index sets which can be obtained by efficient pivots are placed on the unprocessed list and all unbounded efficient edges which we can recognize from our initial efficient basic index set are placed on the third list. We then pick a basic index set from the unprocessed list, put it on the processed list and the process it as above, except that we do not put on the unprocessed list any basic index sets which are already on either of the two first lists. We stop when the unprocessed list is empty.

The “proof” that this algorithm finds all efficient extreme points and all unbounded efficient edges is based upon the theorem (Theorem 9.19 of [16] or Theorem 2.5 of [9]) which states that all efficient basic index sets are connected via efficient pivots. This theorem is however incorrect as seen from the single and multiple objective counter examples given above. As the tableaus in Figures 2 and 3 show, it is not always possible to get from one efficient basic index set to another that is adjacent by an efficient pivot. A correct version of the theorem about the connectedness of all efficient extreme points via efficient pivots is as follows.

**Theorem 2** *Let  $\mathfrak{B}$  be an efficient basic index set for (1) and  $\bar{\mathbf{x}}$  an efficient extreme point. Then starting from  $\mathfrak{B}$  and performing only efficient pivots we can arrive at some basic index set corresponding to  $\bar{\mathbf{x}}$ .*

*Proof:* Since  $\mathfrak{B}$  is an efficient basic index set there exists a vector  $\boldsymbol{\lambda} > \mathbf{0}$  such

that  $\mathfrak{B}$  is optimal basic index set for the problem

$$\begin{aligned} \max \lambda^T \mathbf{C}\mathbf{x} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

and similarly there exists a vector  $\bar{\lambda} > \mathbf{0}$  such that  $\bar{\mathbf{x}}$  is optimal for the corresponding problem. Consider the parametric problem

$$\begin{aligned} \max [\lambda + \theta(\bar{\lambda} - \lambda)]^T \mathbf{C}\mathbf{x} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Solving this problem parametrically for  $0 \leq \theta \leq 1$  starting at  $\theta = 0$ , we change basic index sets at those values of  $\theta$  where some reduced cost coefficient changes sign; i.e., we perform efficient pivots. At  $\theta = 1$  we have a basic index set which is optimal for the problem with objective function  $\bar{\lambda}^T \mathbf{C}\mathbf{x}$ . Let the corresponding extreme point be  $\tilde{\mathbf{x}}$ . If  $\bar{\mathbf{x}}$  is non-degenerate, then  $\bar{\mathbf{x}} = \tilde{\mathbf{x}}$  and we are finished. If not, then carry out the procedure described in Section 2 for finding all optimal basic solutions in this problem. Since we do this by working with the reduced system, we are performing only efficient pivots and since we get all optimal basic solutions we will get  $\bar{\mathbf{x}}$  as desired.

Now let us examine the Evans-Steuer algorithm in the light of Theorem 2. Let  $\mathfrak{B}$  be the initial efficient basic index set and let  $\bar{\mathbf{x}}$  be an efficient extreme point. There is a basic index set  $\bar{\mathfrak{B}}$  corresponding to  $\bar{\mathbf{x}}$  which may be reached from  $\mathfrak{B}$  by performing efficient pivots and it is clear that  $\bar{\mathfrak{B}}$  will appear on the processed list eventually. Thus at least one basic index set corresponding to each efficient extreme point will be found, as claimed by Evans and Steuer, but this only means that the algorithm (and hence ADBASE) is more efficient than originally thought since fewer redundant basic sets will be found.

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