# An Overview in Graphs on Multiple Objective Programming

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**Abstract**

One of the keys to getting one's arms around multiple objective programming is to understand its geometry. With this in mind, the purpose of this paper is to function as a short tutorial on multiple objective programming that is accomplished maximally with graphs, and minimally with text.

## 1 Introduction

Consider the multiple objective program

$$\begin{aligned} &max\,\{f_1(x) = z_1\} \\ &\vdots \\ &max\,\{f_k(x) = z_k\} \\ &s.t. \quad x \in S \end{aligned} \tag{1}$$

where $k$ is the number of objectives, the $z_i$ are *criterion values*, and $S$ is the feasible region in *decision space*. Let $Z \subset R^k$ denote the feasible region in *criterion space* where $z \in Z$ iff there exists an $x \in S$ such that $z = (f_1(x), \ldots, f_k(x))$. Let $K = \{1, \ldots, k\}$. *Criterion vector* $\bar{z} \in Z$ is *nondominated* iff there does not exist another $z \in Z$ such that $z_i \geq \bar{z}_i$ for all $i \in K$ and $z_i > \bar{z}_i$ for at least one $i \in K$. The set of all nondominated criterion vectors is designated $N$ and is called the *nondominated set*. A point $\bar{x} \in S$ is *efficient* iff its criterion vector $\bar{z} = (f_1(\bar{x}), \ldots, f_k(\bar{x}))$ is nondominated. The set of all efficient points is designated $E$ and is called the *efficient set*.

Letting $U : R^k \to R$ be the utility function of the decision maker (DM), any $z^0 \in Z$ that maximizes $U$ over $Z$ is an *optimal criterion vector* and any $x^0 \in S$ such that $(f_1(x^0), \ldots, f_k(x^0)) = z^0$ is an *optimal solution*. We are interested in the efficient and nondominated sets because if $U$ is *coordinatewise increasing* (i.e., more is always better that less of each $f_i(x)$), $z^0 \in N$ and any *inverse image*

$x^0 \in E$. Thus, instead of searching all of $Z$, we need only find the best criterion vector in $N$ to locate an optimal solution to a multiple objective problem. To overview the nature of multiple objective programming problems and methods for solving them, we have the following tutorial topics:

1. Decision Space vs. Criterion Space
2. Ideal Way to Solve a Multiple Objective Program?
3. Graphically Detecting Nondominated Criterion Vectors
4. Reference Criterion Vectors
5. Size of the Nondominated Set
6. Weighted Tchebycheff Metrics
7. Points on Smallest Intersecting Contours
8. Lexicographic Weighted Tchebycheff Sampling Program
9. T-Vertex $\lambda$-Vectors
10. Wierzbicki's Aspiration Criterion Vector Method
11. Tchebycheff Method
12. Why Not Weighted-Sums Approach?
13. Other Interactive Procedures

## 2 Tutorial Topics

**1. Decision Space vs. Criterion Space (Figs. 1–2).** Whereas single objective programming is typically studied in decision space, multiple objective programming is mostly studied in criterion space. To illustrate, consider the two-objective multiple objective linear program (MOLP) of Fig. 1 in which $c^1 = (3, 1, -2)$ and $c^2 = (-1, 2, 0)$ are the gradients of the objective functions, and $S$, the feasible region in decision space, is the unit cube in $R^3$. Fig. 2 shows the feasible region $Z$ in criterion space in which the eight $z^i$ are the images of the eight extreme points of $S$. Note that (a) not all extreme points of $S$ map into extreme points of $Z$, (b) $Z$ is at most of dimensionality $k$, and (c) $Z$ is not necessarily confined to the nonnegative orthant.

**2. Ideal Way to Solve a Multiple Objective Program? (Fig. 3).** While one might consider maximizing $U$ over $Z$ as in Fig. 3 to be an ideal way to solve a multiple objective program, this approach does not work in practice because (a) of the impossibility of obtaining $U$, (b) the approach does not give the user a feel for the nondominated set $N$, and (c) the approach does not allow for "learning" during the course of the solution process. Therefore, without *explicit* knowledge of $U$, the emphasis is on methods that only rely on *implicit* information (e.g., extracting from the DM answers to questions such as which in a group of solutions is the most preferred, or which objectives can be relaxed to enable more achievement in others).

**3. Graphically Detecting Nondominated Criterion Vectors (Figs. 4–7).** Let $R^+$ be the nonnegative orthant in $R^k$. To determine graphically whether a $\bar{z} \in Z$ is dominated or nondominated, translate $R^+$ to $\bar{z}$. This forms

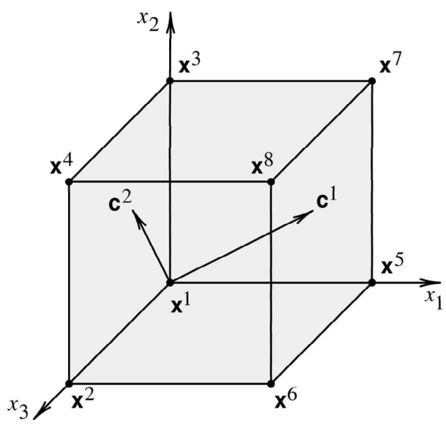


Figure 1: Feasible $S$ in Decision Space

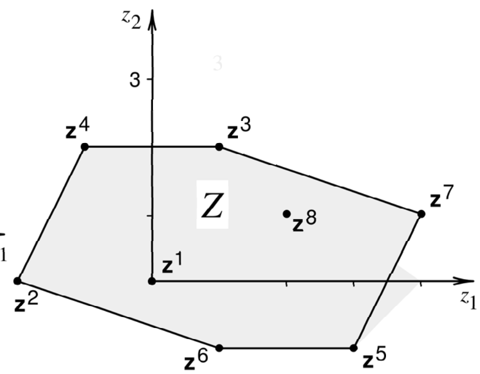


Figure 2: Feasible $Z$ in Criterion Space

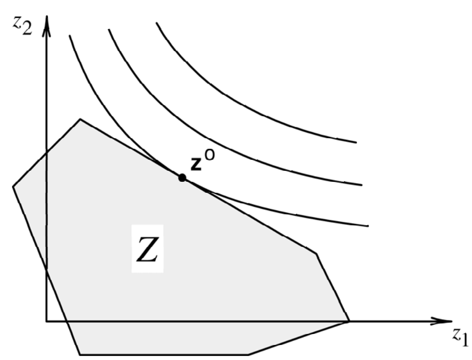


Figure 3: Maximizing $U$ over $Z$

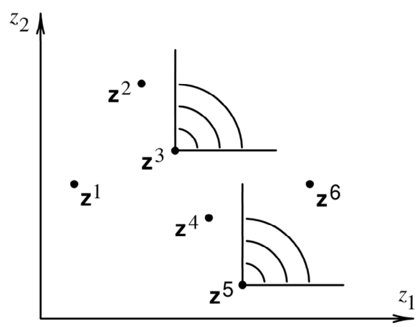


Figure 4: Nondominance in an Integer Case

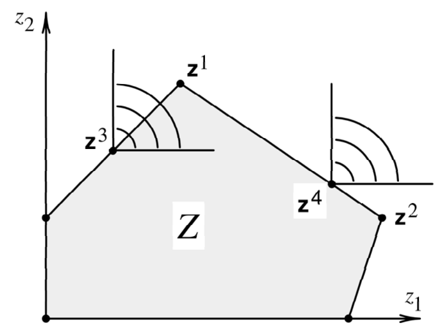


Figure 5: Nondominance in an MOLP

the set called the *translated nonnegative orthant at* $\bar{z}$, designated $\{\bar{z}\} \oplus R^+$ where $\oplus$ denotes *set addition.*

**Theorem 1** *Criterion vector* $\bar{z} \in Z$ *is nondominated iff* $(\{\bar{z}\} \oplus R^+) \cap Z = \{\bar{z}\}$.

In other words, a $\bar{z} \in Z$ is nondominated if and only if, aside from $\bar{z}$, no other points reside in the *intersection* between the nonnegative orthant translated to $\bar{z}$ and $Z$ except for $\bar{z}$ itself. In the integer multiple objective program of Fig. 4 in which $Z = \{z^i \mid 1 \leq i \leq 6\}$, $N = \{z^2, z^3, z^6\}$. For instance, $z^3$ is nondominated because if there were a $z^i \in Z$ dominating $z^3$, it would have to be in the translated nonnegative orthant. On the other hand, $z^5$ is dominated because $z^6$ is in the intersection. In the MOLP of Fig. 5, $N = \text{bls}[z^1 \to z^2]$, where the notation designates the *boundary line segment* in the *clockwise* direction from $z^1$ to $z^2$. For instance, $z^4$ is nondominated because the intersection is empty other than for $z^4$. On the other hand, $z^3$ is dominated. With regard to the extreme points of $Z$, of the five, two are nondominated. In the multiple objective program of Fig. 6, $N = \{z^1\} \cup \text{bls}(z^2 \to z^3]$, where the left parenthesis in the "bls" signifies that $z^2$ is an *open* endpoint. This occurs because $z^2$ is dominated by $z^1$ (i.e., $z^1 \in (\{z^2\} \oplus R^+)$). In Fig. 7, $N = \text{bls}[z^1 \to z^2) \cup \text{bls}[z^3 \to z^4] \cup \text{bls}(z^5 \to z^6]$.

**4. Reference Criterion Vectors (Fig. 8).** Let $K = 1, \ldots, k$ and let $z^{ref} \in R^k$ be a *reference criterion vector* whose components are given by

$$z_i^{ref} = max\,\{f_i(x) \mid x \in S\} + \epsilon_i$$

where the $\epsilon_i$ need only be small positive values. An often convenient scheme is to use values for $\epsilon_i$ that raise each $z_i^{ref}$ to the smallest integer greater than $max\,\{f_i(x) \mid x \in S\}$ as in Fig. 8. A $z^{ref}$ serves two purposes. One is to define the *domain* of the problem $D = \{z \in R^k \mid z_i \leq z_i^{ref}\}$, the region to the "lower left" of the dashed lines in which everything relevant happens. The other, since $z^{ref}$ dominates all points in $N$, is to function as point from which *downward probes* can be made to sample $N$.

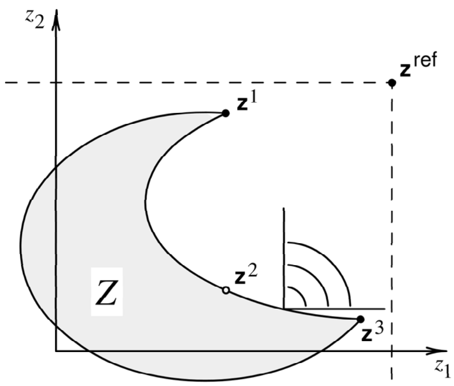


Figure 6: Nondominance in a Nonlinear Case

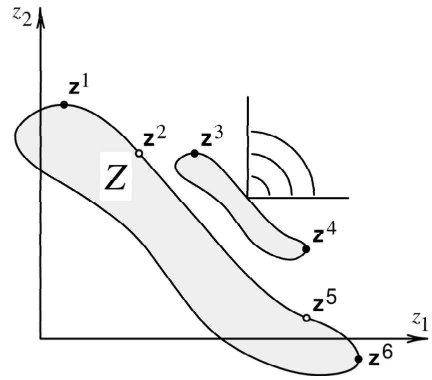


Figure 7: Nondominance in Another Nonlinear Case

**5. Size of the Nondominated Set (Table 1).** Now a few facts about the nondominated set. While $N$ is always connected in the linear case, $N$ can be disconnected in integer and nonlinear cases. Also, $N$ is always a portion of the surface of $Z$. However, since the portion is often quite large, finding a most preferred point in $N$ is often not a trivial task. From computational results gleaned from Steuer [10], Table 1 indicates the numbers of nondominated extreme points typically possessed by MOLPs of different sizes. While $N$ grows with the number of variables and constraints, it grows most substantially with the number of objectives.

| MOLP Problem Size | Typical Number of Nondominated Extreme Points |
|---|---|
| $3 \times 30 \times 25$ | 100 |
| $3 \times 50 \times 50$ | 500 |
| $4 \times 30 \times 50$ | 1000 |
| $5 \times 40 \times 25$ | 3000 |

Table 1: Nondominated Extreme Point Indications of Size of $N$

**6. Weighted Tchebycheff Metric (Fig. 9).** To compute a distance between a $z \in Z$ and $z^{ref}$, it is often useful to employ a $\lambda$-*weighted Tchebycheff metric*

$$\| z - z^{ref} \|_{\infty}^{\lambda} = \max_{i \in K} \{ \lambda_i \mid z_i - z_i^{ref} \mid \}$$

where (a) $\lambda \in \Lambda = \{\lambda \in R^k \mid \lambda_i \in (0,1), \sum_{i \in K} \lambda_i = 1\}$ and (b) associated with each $\lambda$-weighted Tchebycheff metric is a *probing ray* emanating from $z^{ref}$ in the downward direction $-(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k})$. The *contours* (of points in $R^k$ equidistant

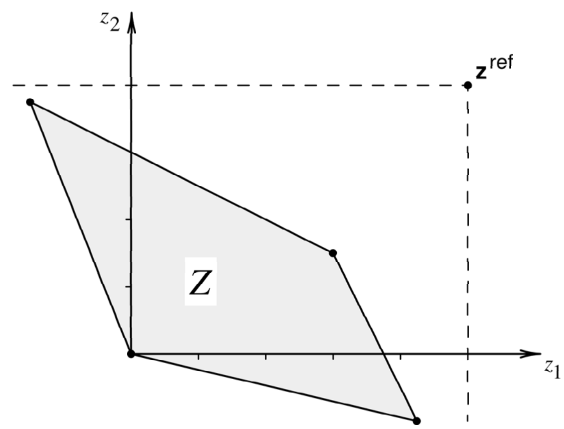


Figure 8: Construction of a $z^{ref}$ Reference Point

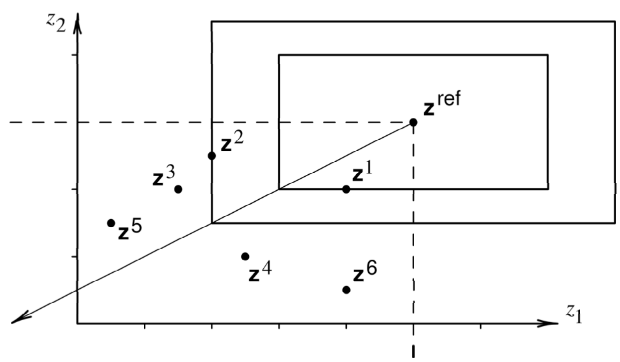


Figure 9: Determining Points Closest to $z^{ref}$

from $z^{ref}$) of a given $\lambda$-weighted Tchebycheff metric form a family of rectangles centered at $z^{ref}$. Moreover, in the domain $D$ of a problem, the vertices of all of the rectangles lie along the probing ray. In Fig. 9 with the probing ray in the direction $-(2,1)$, the rectangles are contours of the $\lambda = (\frac{1}{3}, \frac{2}{3})$ weighted Tchebycheff metric. With this metric, $z^1$ is closest to $z^{ref}$ because it lies on the smallest rectangle, $z^2$ is the next closest, and so forth. Note that by changing the $\lambda$-vector, we change the direction of the probing ray and thus the shape of the rectangular contours.

**7. Points on Smallest Intersecting Contours (Figs. 10–12).** In Fig. 10, $N = \text{bls}[z^1 \to z^2]$. With the probing ray as drawn, it is only necessary to show the portion of the smallest intersecting rectangular contour in $D$ to see that $z^3$ is the point in $Z$ closest to $z^{ref}$. Because the portion of any rectangular contour in $D$ also portrays a translated nonnegative orthant, we further observe

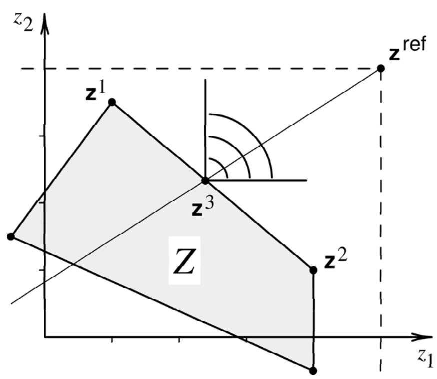


Figure 10: Point Ecountered with Vertex

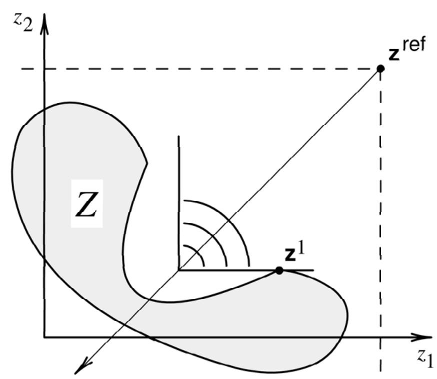


Figure 11: Point Encountered on Side

that $z^3$ is nondominated. Several questions now arise. What happens if

(a) the point on the smallest intersecting contour does not occur at the vertex of the contour as in Fig. 11?

(b) there is more than one point on the smallest intersecting contour as in Fig. 12? Are they all nondominated?

As long as the point on the smallest intersecting contour is unique, by Theorem 2 (a generalization of Theorem 1), the point is nondominated.

**Theorem 2** *Let $\bar{z} \in Z$ and let $z^v \in R^k$ function as the vertex of a translated nonnegative orthant such that $(\{z^v\} \oplus R^+) \cap Z = \{\bar{z}\}$. Then $\bar{z}$ in nondominated.*

In answer to (b), let $Z_\lambda \subset Z$ be the set of points on the smallest intersecting contour. By Theorem 3, at least one point in $Z_\lambda$ is nondominated.

**Theorem 3** *Let $\lambda \in \Lambda$. Then $\bar{z}$ is nondominated if it is a point in $Z_\lambda$ that is closest to a $z^{ref}$ according to a $L_1$-metric.*

In Fig. 12 with $Z_\lambda = \text{bls}[z^2 \to z^3]$, $z^3$ is seen to be nondominated as it is the point in $Z_\lambda$ that minimizes the sum of the coordinate deviations between it and $z^{ref}$, or in other words, $z^3$ is the point that solves

$$\min_{z \in Z_\lambda} \left[ \sum_{i=1}^{k} (z_i^{ref} - z_i) \right]$$

**8. Lexicographic Weighted Tchebycheff Sampling Program .** With regard to the ability of a weighted Tchebycheff metric to *generate* a nondominated criterion vector, we have a two-stage process. In the first stage, we compute $Z_\lambda$, the set of points in $Z$ closest to $z^{ref}$ according to the $\lambda$-weighted

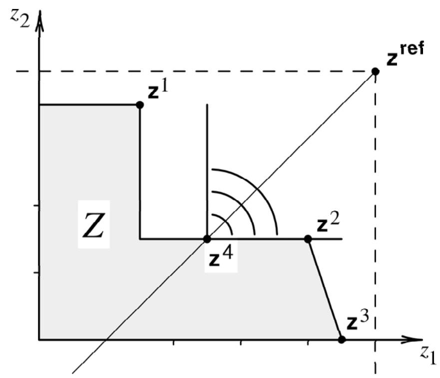


Figure 12: Ties on Smallest Intersecting Contour

Tchebycheff metric. If $Z_\lambda$ is a singleton set, the point in $Z_\lambda$ is the nondominated point generated. If $Z_\lambda$ contains more than one point, the second stage is called into action to compute, as the nondominnated point generated, a point in $Z_\lambda$ closest to $z^{ref}$ according to the $L_1$-metric. Incorporating this geometry into an optimization problem, we have the two-stage *lexicographic weighted Tchebycheff sampling* program for

$$
\begin{aligned}
\text{lex min}\{\, \alpha, \quad & \sum_{i=1}^{k}(z_i^{ref} - z_i)\} \qquad (2)\\
s.t. \quad \alpha \geq \lambda_i(z_i^{ref} &- z_i)\\
f_i(x) &= z_i\\
x &\in S\\
0 \leq \alpha \in R,\ z \in R^k &\text{unrestricted}
\end{aligned}
$$

generating, not only a nondominated criterion vector, but an inverse image (i.e., an efficient point) from $S$ that goes with it. Note that the first optimization stage minimizes the scalar $\alpha$ to implement the $\lambda$-weighted Tchebycheff metric, and that the second stage, when invoked, minimizes $\sum_{i\in K}(z_i^{ref} - z_i)$ to implement the $L_1$-metric. It is called a "sampling" program because in this way, with a group of dispersed representatives from $\Lambda$, and solving (2) for each of them, we have a strategy for sampling points from $N$.

**9. T-Vertex $\lambda$-Vectors**. Note that in some cases, many different $\lambda$-vectors could cause the lexicographic weighted Tchebycheff sampling program to generate the same nondominated point. However, out of each such set, one is special and it is called the *T-vertex* $\lambda$-vector. The T-vertex $\lambda$-vector is the one that causes the smallest intersecting contour to hit the nondominated point in question head-on with its vertex. For a given $z \in N$ and $z^{ref}$, the coordinates of a

T-vertex $\lambda$-vector are given by

$$\lambda_i = \frac{1}{z_i^{ref} - z_i} \left[ \sum_{j \in K}^{k} \frac{1}{z_j^{ref} - z_j} \right]^{-1} \tag{3}$$

**10. Wierzbicki's Aspiration Criterion Vector Method (Figs. 13-14).** As an introduction to the world of interactive procedures of multiple objective programming, we first discuss Wierzbicki's Aspiration Criterion Vector method [12]. The purpose of the procedure is to find at each iteration $h$ the point $z^{(h)} \in N$ that is closest to that iteration's *aspiration criterion vector* $q^{(h)} < z^{ref}$, where $q^{(h)}$'s purpose is to capture the DM's criterion value hopes, expectations, and aspirations of the moment.

1. $h = 0$. Establish a $z^{ref} \in R^k$.
2. $h = h + 1$. Specify a $q^{(h)}$ aspiration criterion vector.
3. Compute the $\lambda^{(h)} \in \Lambda$ that causes the probing ray to pass through $q^{(h)}$. Equation (3) can be used by substituting $q^{(h)}$ for $z^{ref}$.
4. Using $\lambda^{(h)}$ in (2), obtain the nondominated point $z^{(h)}$ closest to $q^{(h)}$ as computed by the lexicographic weighted Tchebycheff sampling program.
5. If after examining $z^{(h)}$ the DM decides to specify another aspiration criterion vector, go to Step 2. Otherwise, terminate with $z^{(h)}$ as the *final solution.*

Consider the feasible region $Z$ in Figs. 13 and 14 with $z^{ref} = (5, 4)$. On the first iteration, let the DM's aspiration criterion vector be $q^{(1)} = (3\frac{1}{2}, 3\frac{1}{2})$. Then $\lambda^{(1)} = (\frac{1}{4}, \frac{3}{4})$ and the nondominated point generated by (2) is $z^{(1)}$. Assuming the DM wishes to continue, the DM specifies a $q^{(2)} = (3, 1)$. Then $\lambda^{(2)} = (\frac{3}{5}, \frac{2}{5})$ and the nondominated point generated by (2) is $z^{(2)}$. This iteration is interesting because, recognizing that $q^{(2)}$ is dominated, the method produces a superior nondominating point. Lacking a formal stopping criterion, the method continues as long as the DM is willing to specify new aspiration criterion vectors.

**11. Tchebycheff Method (Figs. 15–16).** Instead of generating only one solution at each iteration, the Tchebycheff Method conducts *multiple probes* by sampling each in a sequence of progressively small subsets of $N$. Letting $P$ be the number of solutions presented to the DM at each iteration, we begin by selecting $P$ dispersed $\lambda$-vectors from $\Lambda^{(1)} = \Lambda$. Then the lexicographic weighted Tchebycheff program (2) is solved for each of the $\lambda$-vectors. From the $P$ resulting nondominated criterion vectors, the DM selects the most preferred, designating it $z^{(1)}$. Now, about the T-vertex $\lambda$-vector defined by $z^{(1)}$ and $z^{ref}$, a reduced subset $\Lambda^{(2)} \subset \Lambda$ is centered. Then $P$ dispersed $\lambda$-vectors are selected from $\Lambda^{(2)}$

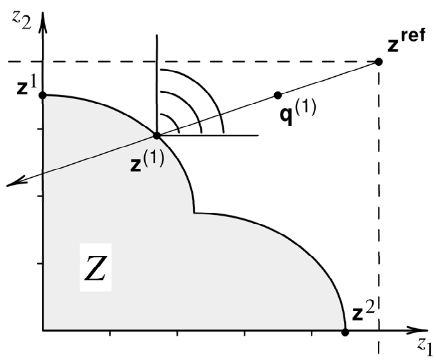


Figure 13: Wierzbicki 1-st Iteration

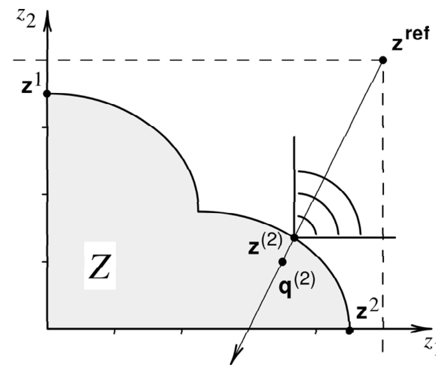


Figure 14: Wierzbicki 2-nd Iteration

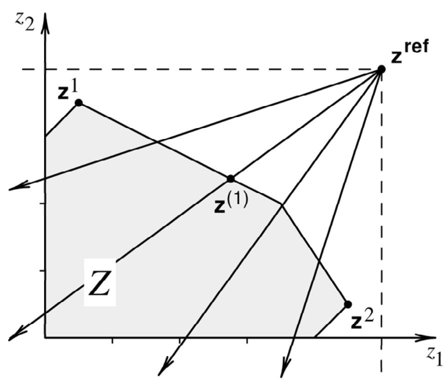


Figure 15: Tchebycheff 1-st Iteration

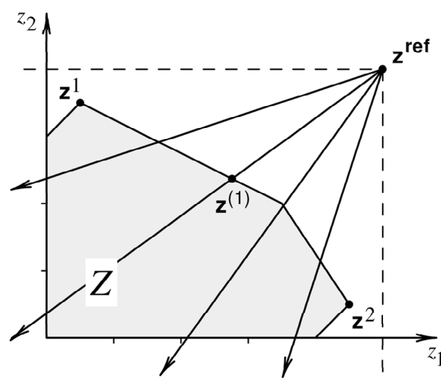


Figure 16: Tchebycheff 2-nd Iteration

and (2) is solved for each of them. From the $P$ resulting nondominated criterion vectors, the DM selects the most preferred, designating it $z^{(2)}$. Now, about the T-vertex $\lambda$-vector defined by $z^{(2)}$ and $z^{ref}$, a further reduced subset $\Lambda^{(3)} \subset \Lambda$ is centered. Then $P$ dispersed $\lambda$-vectors are selected from $\Lambda^{(3)}$, and so forth.

**11. Why Not Weighted-Sums Approach?** One might ask why not to considered assigning a $\lambda_i$-weight to each objective function and solve the *weighted-sums* program

$$\max\{\sum_{i \in K}^{k} \lambda_i f_i(x) \mid x \in S\}$$

The reason is that the weighted-sums program only computes points that support hyperplanes and is unable to compute *unsupported* nondominated criterion vectors. A $\bar{z} \in N$ is unsupported if and only if there exists a convex combination of other nondominated criterion vectors that dominates $\bar{z}$. Otherwise, $\bar{z}$

is *supported*. Let $N^u$ and $N^s$ designate the sets of unsupported and supported nondominated criterion vectors, respectively. For example in Fig. 4, $N^u = \{z^3\}$ and $N^s = \{z^2, z^6\}$. In Fig. 6, $N^u = \text{bls}(z^2 \to z^3)$ and $N^s = \{z^1, z^3\}$, and so forth. If an unsupported point were optimal, the weighted-sums program would be unable to compute it. In contrast, the lexicographic weighted Tchebycheff program can compute *any* nondominated criterion vector without reservation.

**12. Other Interactive Procedures.** In addition to the Aspiration Criterion Vector and Tchebycheff methods, there are other procedures of multiple objective programming including STEM [1], Global Shooting [2], TRIMAP [3], Light Beam Search [5], Pareto Race [6], Bi-Reference Procedure [7], Fuzzy Satisficing [8], PSI [9], and FFANN [11], among others. While embodying various philosophies, most of these procedures nevertheless use variants of the lexicographic weighted Tchebycheff program [4] to probe $N$ in different ways.

## 3 Conclusion

Because the lexicographic weighted Tchebycheff sampling program and its variants in other procedures can be configured for solution using conventional single criterion mathematical programming software, the procedures of multiple objective programming can generally address problems with as many constraints and variables as in the single criterion case. However, there is a limit to the number of objectives. While problems with up to 5–6 objectives can generally be accommodated, above this number gets us into uncharted territory where future research is needed.

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