Portfolio Selection in the Presence of Multiple Criteria

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1 Introduction

There has been a growing interest in how to incorporate additional criteria beyond "risk and return" into the portfolio selection process. In response, our purpose is to describe the latest in results that have been coming together under the topic of multiple criteria portfolio selection. Starting with a review of conventional portfolio selection from a somewhat different perspective so as better to lead into the topic of multiple criteria portfolio selection, we start from basics as follows.

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In portfolio selection, two vectors are associated with each portfolio. One is used to "define" a portfolio. The other is used to "describe" the portfolio. The vector used to define a portfolio is an *investment proportion vector*. It specifies the proportions of an amount of money to be invested in different securities thereby defining the composition of a portfolio. The length of an investment proportion vector is the number of securities under consideration.

The other of a portfolio's two vectors is a *criterion vector*. A portfolio's criterion vector contains the values of measures used to evaluate the portfolio. For instance, in mean-variance portfolio selection, criterion vectors have two components. One is for specifying the expected value of the portfolio's return random variable. The other is for specifying the variance of the random variable. The idea is that the variance of the random variable is a measure of risk. In reality, investors may have additional concerns.

To accommodate multiple criteria in portfolio selection, we no longer call an "efficient frontier" by that name. Instead we call it a "nondominated frontier" or "nondominated set." Terminologically, criterion vectors are now either *nondominated* or *dominated*. This does not mean that the term efficiency has been discarded. Efficiency is simply re-directed to apply only to investment proportion vectors in the following sense. An investment proportion vector is *efficient* if and only if its criterion vector is nondominated, and an investment proportion vector is *inefficient* if and only if its criterion vector is dominated.

A portfolio selection problem is a *multiple criteria portfolio selection* problem when its criterion vectors have three or more components. In conventional portfolio selection, with criterion vectors of length two, the nondominated set is typically a curved line in two-dimensional space, which when graphed usually has expected value of the portfolio return random variable on the vertical axis and variance (or more commonly, standard deviation) of the same random variable on the horizontal axis. But, when criterion vectors are of length three or more, the nondominated set is best thought of as a surface in higher dimensional space. Because of the increased difficulties involved in computing nondominated surfaces and communicating them to investors, multiple criteria portfolio selection problems can be expected to be much more difficult to solve than the types of problems we are used to seeing in conventional portfolio selection.

Multiple criteria portfolio selection problems normally stem from multiple-argument investor utility functions, but can stem from a single-argument utility function.[1] While portfolio selection problems with criterion vectors of length two are the usual case with single-argument utility functions (when the argument is stochastic), it is possible for a multiple criteria portfolio selection problem to result from a single-argument utility function when the investor's nondominated set is a consequence of three or more measures derived from the same single stochastic argument. An example of this is when a mean-variance portfolio selection problem (which revolves around the single random variable of portfolio return) is extended to take into account additional measures, such as skewness, based upon the same random variable.

Despite the above, we will primarily focus on the more general and interesting cases of multiple criteria portfolio selection problems resulting from multiple-argument utility functions. Beyond the random variable of portfolio return, utility functions can take additional stochastic and deterministic arguments. Additional stochastic arguments might include dividend, liquidity, and excess return over of a benchmark random variables. Deterministic arguments might include the number of securities in a portfolio, turnover, and amount of short selling.

Conventional mean-variance portfolio analysis (described as "modern portfolio analysis" in Elton, Gruber, Brown and Goetzmann (2002)) dates back to the papers of Roy (1952) and Markowitz (1952). In addition to introducing new ways to think about finance, the papers are important because they symbolize different strategies for solving portfolio selection problems. In computing his "safety first" point, Roy's paper symbolizes approaches that attempt to compute directly portfolios whose criterion vectors possess pre-chosen characteristics.

On the other hand, the approach of Markowitz is more reflective. It recognizes that there are likely to be differences among investors. It essentially eschews pre-conceived notions preferring to compute the entire nondominated set first. Then, only after studying the nondominated set should an investor attempt to identify a most preferred criterion vector. Overall, Markowitz's solution approach consists of the following four stages.

1. Compute the nondominated set.
2. Communicate the nondominated set to the investor.
3. Select the most preferred of the points in the nondominated set.

[1] Although we use the term *utility function* throughout, *preference function* or *value function* could just have well been used.

4. Working backwards, identify an investment proportion vector whose image is the nondominated point (i.e., criterion vector) selected in Stage 3.

Under assumptions generally accepted in portfolio selection, these four stages, when properly carried out, will lead to an optimal portfolio of the investor. Because of the widespread acceptance of Markowitz's approach, his name is virtually synonymous with portfolio selection although Markowitz (1999) has tried to see that Roy also receives credit.

Despite the degree to which mean-variance portfolio selection dominates the landscape, there has almost always been a slight undercurrent of multiple objectives in portfolio selection. However, this undercurrent has been becoming more pronounced of late. For instance, from Steuer and Na (2003), the number of papers reported as dealing with multiple criteria in portfolio selection has increased from about 1.5 to 4.5 per year over the period 1973-2000. Such papers can be grouped into three categories.

In the first category we have overview articles such as by Colson and DeBruyn (1989), Spronk and Hallerbach (1997), Bana e Costa and Soares (2001), Hallerbach and Spronk (2002a, 2002b), Spronk, Steuer and Zopounidis (2005), and Steuer, Qi and Hirschberger (2005, 2006a, 2006b).

In the second category, in the spirit of Roy, are articles that attempt to compute directly points on the nondominated surface that possess certain characteristics. Papers in this category include Lee and Lerro (1973), Hurson and Zopounidis (1995), Ballestero and Romero (1996), Dominiak (1997a, 1997b), Doumpos, Spanos and Zopounidis (1999), Arenas Parra, Bilbao Terol and Rodríguez Uría (2001), Ballestero (2002), Bouri, Martel and Chabchoub (2002), Ballestero and Plà-Santamaría (2004), Bana e Costa and Soares (2004), and Aouni, Ben Abdelaziz and El-Fayedh (2006).

In the third category, in the spirit of Markowitz, are articles that attempt to compute, or at least interactively search or sample, the nondominated set before selecting a"final" portfolio. Here, a *final* solution is a portfolio that is either optimal, or sufficiently close to being optimal to terminate the decision process. Contributions in this category include those by Spronk (1981), Konno, Shirakawa and Yamazaki (1993), L'Hoir and Teghem (1995), Chow (1995), Tamiz, Hasham, Jones, Hesni and Fargher (1996), Korhonen and Yu (1997), Yu (1997), Ogryczak (2000), and Xu and Li (2002), Lo, Petrov and Wierzbicki (2003), Ehrgott, Klamroth and Schwehm (2004), Fliege (2004), and Kliber (2005).

The organization of the rest of this article is as follows. Sections 2 and 3 discuss the initially stochastic, and then deterministic, nature of portfolio selection. Section 4 discusses single- and multiple-argument utility functions and shows the natural way multiple-argument utility functions lead to multiple criteria portfolio selection formulations. After a careful study of the mean-variance nondominated frontier in Section 5, the nondominated sets of multiple criteria portfolio selection problems, and issues involved in their computation, are discussed in Section 6. Section 7 closes the article.

2 Initial Stochastic Programming Problem

In its most basic form, the problem of portfolio selection is as follows. Consider a fixed sum of money to be invested in securities selected from a universe of n securities. Let there be a beginning of a holding period and an end of the holding period. Also, let x_i be the proportion of the fixed sum to be invested in the i-th security. Being proportions, the sum of the x_i equals one.

Let r_i denote the random variable for the i-th security's return over the holding period. While the realized values of the r_i are not known until the end of the holding period, it is nevertheless assumed that all means μ_i, variances σ_{ii}, and covariances σ_{ij} of the distributions from which the r_i come are known at the beginning of the holding period.

Letting r_p denote the random variable for the return on a portfolio defined by the r_i and some set of x_i over the holding period, we have

$$r_p = \sum_{i=1}^{n} r_i x_i$$

Under the assumption that investors are only interested in maximizing the uncertain objective of return on a portfolio, the problem of portfolio selection is then to maximize r_p as in

$$\max\{\, r_p = \sum_{i=1}^{n} r_i x_i \} \tag{1}$$
$$\text{s.t.} \quad \mathbf{x} \in S = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ \alpha_i \le x_i \le \omega_i\}$$

where S as above is a typical feasible region. While (1) may look like a linear programming problem, it is not a linear programming problem. Since the r_i are not known until the end of the holding period, but the x_i must be determined at the beginning of the holding period, (1) is a *stochastic* programming problem. For use later, let (1) be called the investor's *initial stochastic programming problem*. As stated in Caballero, Cerdá, Muñoz, Rey and Stancu-Minasian (2001), if in a problem some parameters take unknown values at the time of making a decision, and these parameters are random variables, then the resulting problem is called a stochastic programming problem. Since S is deterministic, (1)'s stochastic nature only derives from random variable elements being present in the objective function portion of the program. Interested readers might also wish to consult Ziemba (2003) for additional stochastic discussions about portfolio selection.

3 Equivalent Deterministic Formulations

The difficulty with a stochastic programming problem is that its solution is not well defined. Hence, to solve (1) requires an interpretation and a decision. The approach taken in the literature (for instance in Stancu-Minasian (1984), Slowinski and Teghem (1990), and Prékopa (1995)) is to ultimately transform the stochastic problem into an *equivalent deterministic problem* for solution. Equivalent deterministic problems typically involve the utilization of some statistical characteristic or characteristics of the random variables in question. For problems with a single stochastic objective as in (1), Caballero, Cerdá, Muñoz, Rey and Stancu-Minasian discuss the following five equivalent deterministic possibilities:

(a) $\max\{E[r_p]\}$
s.t. $\mathbf{x} \in S$

(b) $\min\{Var[r_p]\}$
s.t. $\mathbf{x} \in S$

(c) $\max\{E[r_p]\}$
$\min\{Var[r_p]\}$
s.t. $\mathbf{x} \in S$

(d) $\max\{P(r_p) \geq u\}$ for some chosen level of u
s.t. $\mathbf{x} \in S$

(e) $\max\{u\}$
s.t. $P(r_p \geq u) \geq \beta$ for some chosen level of β
$\mathbf{x} \in S$

If there is a question about how any of the above can be deterministic, recall that from the previous section all means μ_i, variances σ_{ii}, and covariances σ_{ij} of the r_i are assumed to be known at the beginning of the holding period. But with a list of choices, how is one to know which should replace (1) for a given investor? At this point it is illuminating to take a step back and delve into the rationale that leads from the investor's initial stochastic programming problem to equivalent deterministic possibilities **(a)** to **(e)**.

Early seventeenth century mathematicians assumed that a gambler would be indifferent between receiving the uncertain outcome of a gamble and receiving in cash its expected value. In the context of portfolio selection, the gambler would be an investor, the gamble would be the return on a portfolio, and the *certainty equivalent* would be

$$CE = E[r_p]$$

Given that an investor would want to maximize the amount of cash received for certain, this rationale leads directly to equivalent deterministic possibility **(a)**. However, Bernoulli (1738) discovered what has become known as the St. Petersburg paradox.[2] A coin is tossed until it lands "heads." The gambler receives one ducat if it lands "heads" on the first throw, two ducats if it first lands "heads" on the second throw, four ducats if it first lands "heads" on the third throw, and so on (2^{h-1} ducats on the h-th throw). The expected value of the gamble is infinite, but in reality many gamblers would be willing to accept only a small number of ducats in exchange for the gamble. Hence, Bernoulli suggested not to compare cash outcomes, but to compare the "utilities" of cash outcomes. With the utility of a cash outcome given by a $U : \mathbb{R} \to \mathbb{R}$, we thus have

$$U(CE) = E[U(r_p)]$$

That is, the utility of *CE* equals the expected utility of the uncertain portfolio return.

[2]Because it was published in the *Commentaries from the Academy of Sciences of St. Petersburg.*

With an investor wishing to maximize $U(CE)$, this leads to the problem of Bernoulli's principle of maximum expected utility

$$\begin{aligned} &\max\{E[U(r_p)]\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{2}$$

With U obviously increasing with r_p, this means that any $\mathbf{x}$ that solves (2) solves (1), and vice versa. Although Bernoulli's maximum expected utility problem (2) is a deterministic equivalent to (1), we call it an equivalent "undetermined" deterministic problem. This is because it is not fully *determined* in that it contains unknown utility function parameters and cannot be solved in its present form. However, with investors assumed to be *risk averse*, (i.e., the expected value $E[r_p]$ is always preferred over the uncertain outcome r_p), we at least know that in (2) U is concave.

Two schools of thought have evolved for dealing with the undetermined nature of U. One, in the spirit of Roy, involves attempting to ascertain aspects of an investor's preference structure for the purpose of using them to solve (2) directly for an optimal portfolio. The other, in the spirit of Markowitz, involves parameterizing U and then attempting to solve (2) for all possible values of its unknown parameters. With this at the core of contemporary portfolio theory, Markowitz considered a parameterized *quadratic* utility function[3]

$$U(x) = x - (\lambda/2)x^2 \tag{3}$$

Since $U(x)$ above is normalized such that $U(0) = 0$ and $U'(0) = 1$, this leaves exactly one parameter λ, the *coefficient of risk aversion.* By this parameterization, Markowitz showed that precisely all potentially maximizing solutions of the equivalent "undetermined" deterministic problem (2) for a risk averse investor can be obtained by solving equivalent deterministic possibility **(c)**

$$\begin{aligned} &\max\{E[r_p]\} \\ &\min\{Var[r_p]\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned}$$

[3]There is an anomaly with quadratic utility functions since they decrease from a certain point on. Instead of quadratic utility, an alternative argument (not shown) leading to the same result can be made by assuming that U is concave and increasing, and that $\mathbf{r} = (r_1, \ldots, r_n)$ follows a multinormal distribution.

for all $\mathbf{x} \in S$ from which it is not possible to increase expected portfolio return without increasing portfolio variance, or decrease portfolio variance without decreasing expected portfolio return. In accordance with terminology introduced earlier, the set of all such $\mathbf{x}$-vectors constitutes the *efficient set* (in investment proportion space) and the set of all images of the efficient points constitutes the *nondominated set* (in criterion space). Thus, with U as in (3), **(c)** is the most appropriate equivalent deterministic problem from among the five. Note that with respect to the extreme values of $\lambda = 0$ (risk neutrality) or $\lambda \to \infty$ (extreme risk aversion), we obtain possibilities **(a)** or **(b)**, respectively, as special cases of **(c)**. It should be noted that since the limit function of U does not exist for $\lambda \to \infty$, **(b)** is not directly obtained as an expected utility maximizing solution. It is only obtained as the limit of expected utility solutions for increasing risk aversion.

Should we consider another extreme situation in which

$$U(x) = \begin{cases} 1, & c + \varepsilon \leq x \\ (x - c)/\varepsilon & c \leq x < c + \varepsilon, \\ 0, & x < c \end{cases}$$

with an unknown parameter c and an $\varepsilon > 0$, we could observe that

$$P(r \geq c) \geq E[U(r)] \geq P(r \geq c + \varepsilon)$$

For a continuous random variable r, we obtain $E[U(r)] = P(r \geq c)$ for $\varepsilon \to 0$, which would lead to candidates **(d)** and **(e)**. For instance, let c be the risk-free rate of return. Then candidate **(d)** would mean that the probability to receive at least the risk-free rate of return on a portfolio is maximized. If $\mathbf{r} = (r_1, \ldots, r_n)$ follows a multinormal distribution, in the case of c equalling the risk-free rate, solving **(d)** then yields Roy's "safety first" portfolio. Again, it should be noted that **(d)** and **(e)** are not obtained as expected utility maximizing solutions[4], but only as the limit of expected utility solutions for an increasing focus on c.

Although not mentioned in Caballero, Cerdá, Muñoz, Rey and Stancu-Minasian, a sixth equivalent deterministic possibility stemming from (1) is

(f) $\max\{E[r_p]\}$

[4]While the limit function of U does exist for $\varepsilon \to 0$, it contradicts the Archimedean axiom of von Neumann and Morgenstern (1947), i.e. the function is discontinuous.

$$
\begin{aligned}
&\min\{Var[r_p]\} \\
&\max\{Skew[r_p]\} \\
&\text{s.t.} \quad \mathbf{x} \in S
\end{aligned}
$$

where *Skew* stands for skewness. With criterion vectors of length three, **(f)** is a multiple criteria portfolio selection problem. This formulation is probably the only multiple criteria formulation that is not totally unfamiliar to conventional portfolio selection as a result of the interest taken in skewness by authors such as Stone (1973), Konno and Suzuki (1995), Chunhachinda, Dandapani, Hamid and Prakash (1997), Prakash, Chang and Pactwa (2003). However, we will not dwell on **(f)** as this formulation, as a result of the severe nonlinearities of its third criterion, has not gained much traction in practice. Instead, we will concentrate on the newer types of multiple criteria portfolio selection problems that have begun to appear as a result of the more sophisticated purposes of many investors.

4 Portfolio Selection with Multiple-Argument Utility Functions

Whereas multiple criteria formulations are little more than a curiosity in conventional portfolio selection, multiple criteria formulations are mostly appropriate when attempting to meet the modeling needs of investors with multiple-argument utility functions. Two situations in which multiple-argument utility functions are likely to occur are as follows.

One is that in addition to portfolio return, an investor has other considerations, such as to maximize social responsibility or to minimize the number of securities in a portfolio, that are also important to the investor. That is, instead of being interested in solely maximizing the stochastic objective of portfolio return, the investor can be viewed as being interested in optimizing some combination of several stochastic and several deterministic objectives.

A second situation in which a multiple-argument utility function might pertain is when an investor is unwilling to accept the assumption that all means μ_i, variances σ_{ii}, and covariances σ_{ij} can be treated as known at the beginning of the holding period. In response, an investor might wish to monitor the construction of his or her portfolio with the help of additional measures such as dividends, growth in sales, amount invested in R&D, and so forth, to guard against relying on any single measure that might have imperfections associated with it.

Let z_1 be alternative notation for r_p. Then, a list of z_i criterion values, from which arguments might be selected to staff an investor's multiple-argument utility function, is as follows.

max{ z_1 = portfolio return}
max{ z_2 = dividends}
max{ z_3 = growth in sales}
max{ z_4 = social responsibility}
max{ z_5 = liquidity}
max{ z_6 = portfolio return over that of a benchmark}

max{ z_7 = amount invested in R&D}

min{ z_8 = deviations from asset allocation percentages}
min{ z_9 = number of securities in portfolio}
min{ z_{10} = turnover (i.e., costs of adjustment)}
min{ z_{11} = maximum investment proportion weight}
min{ z_{12} = amount of short selling}
min{ z_{13} = number of securities sold short}

Of course, other z_i can be imagined. Note the differences between first and last six of the z_i. For the first six, it is not possible to know the realized values of the z_i until the end of the holding period. Depending in turn upon random variables associated with each of the n securities, these z_i, like z_1, are themselves random variables. Thus, the first six are stochastic objectives.

For the last six z_i, the actual values of these z_i, for any investment proportion vector $\mathbf{x}$, are available at the beginning of the holding period. For example, for any investment proportion vector $\mathbf{x}$, z_9 is given by the number of nonzero components in $\mathbf{x}$. With the last six z_i known in this way at the beginning of the holding period, they are deterministic objectives.

As for z_7 in the middle, it is an example of a measure that could be argued either way. It could be argued that only the most recent amounts invested in R&D are relevant to the situation at the end of the holding period, thus enabling the objective to be treated deterministically.

One might ask why can't extra objectives be handled by means of constraints? The difficulty is in the setting of the right-hand sides of the constraints. In general, for a model to produce a mean-variance nondominated frontier that contains the criterion vector of an optimal portfolio, one would need to know the optimal value of each objective modeled as a constraint prior to computing the frontier. It is not likely that this would be possible in many situations.

With z_1 almost certainly an argument of every investor's utility function, additional arguments depend upon the investor. For instance, one investor's set of arguments might consist of $\{z_1, z_2, z_{10}\}$, and another's might consist of $\{z_1, z_5, z_7, z_8, z_{11}\}$. The point is that all investors need not be the same. If we let k be the number of selected objectives, in the case of the first investor, $k = 3$, and in the case of the second investor, $k = 5$. Of course, a conventional mean-variance investor's set of arguments would only be $\{z_1\}$ in which case $k = 1$.

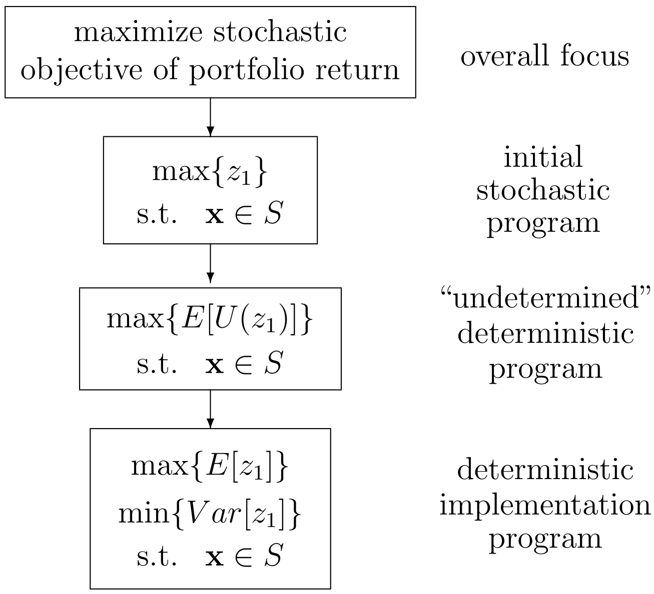

Figure 1: Hierarchical structure of the overall focus, initial stochastic program, equivalent "undetermined" deterministic program, and equivalent deterministic implementation program of conventional portfolio selection.

The differences between conventional portfolio selection and multiple criteria portfolio selection are highlighted in Figures 1 and 2. At the top of each, as in Saaty's Analytic Hierarchy Process (1999), is the investor's *overall focus*. In Figure 1, the overall focus is to maximize the portfolio return random variable. In Figure 2, the overall focus is to optimize some combination of stochastic and deterministic objectives. In the second box of each is the investor's initial stochastic programming problem. Note that the initial stochastic programming problem in Figure 2 reflects the multiple stochastic and deterministic objectives involved in the investor's overall focus and hence is a *multiobjective* stochastic program. As for notation in the second, third and fourth boxes of Figure 2, η specifies the number of stochastic objectives of concern and $D_{i_{\eta+1}}(\mathbf{x})$ represents the first of the $k-\eta$ deterministic objectives of concern. For instance, if $D_{13}(\mathbf{x})$ were included, then $D_{13}(\mathbf{x})$ would represent a function that returns the number of negative x_i.

optimize some combination of stochastic and deterministic objectives	overall focus
$\max\{z_{i_1}\}$ $\vdots$ $\min\{D_{i_{\eta+1}}(\mathbf{x})\}$ $\vdots$ s.t. $\mathbf{x} \in S$	initial (multiobjective) stochastic program
$\max\{E[U(z_{i_1}, \ldots, z_{i_\eta}, z_{i_{\eta+1}}, \ldots, z_{i_k})]\}$ s.t. $\mathbf{x} \in S$	"undetermined" deterministic program
$\max\{E[z_{i_1}]\}$ $\min\{Var[z_{i_1}]\}$ $\vdots$ $\min\{D_{i_{\eta+1}}(\mathbf{x})\}$ $\vdots$ s.t. $\mathbf{x} \in S$	deterministic implementation program

Figure 2: Hierarchical structure of the overall focus, initial (multiobjective) stochastic program, equivalent "undetermined" deterministic program, and equivalent deterministic implementation program of multiple criteria portfolio selection

In the third box of Figure 2 is the equivalent "undetermined" deterministic problem

$$\begin{aligned} \max\{E[U(z_{i_1},\ldots,z_{i_\eta},z_{i_{\eta+1}},\ldots,z_{i_k})]\} \\ \text{s.t.} \quad \mathbf{x}\in S \end{aligned} \tag{4}$$

which shows the multiple-argument utility function that follows from the investor's overall focus. Employing a mean-variance pair for each stochastic argument of the utility function, we have the equivalent deterministic implementation program of the bottommost box. We use the term "implementation" because this is the actual deterministic problem that is implemented. Note that all deterministic objectives of the initial (multiobjective) stochastic program are repeated unchanged in the equivalent deterministic implementation program.

As a practical matter, for stochastic objectives in which variation is small or not of noteworthy importance, it may be possible to represent them in the equivalent deterministic implementation program of the bottommost box of Figure 2 by **(a)** instead of **(c)**. This would be very advantageous when possible. For example, suppose an investor's set is $\{z_1, z_2, z_5\}$. Since these objectives are linear in the portfolio weights. the investor's initial (multiobjective) stochastic program would be

$$\begin{aligned} \max\{\, z_1 &= \sum_{j=1}^{n} r_j x_j\} \\ \max\{\, z_2 &= \sum_{j=1}^{n} d_j x_j\} \\ \max\{\, z_5 &= \sum_{j=1}^{n} \ell_j x_j\} \\ \text{s.t.} \quad & \mathbf{x}\in S \end{aligned}$$

in which d_j is the random variable for the dividends, and ℓ_j is the random variable for the liquidity, of the j-th security. Should variations in portfolio dividends and portfolio liquidity be much less important than variations in portfolio return, then it may well be acceptable to use **(a)** instead of **(c)** for each of dividends and liquidity. Then the resulting equivalent deterministic implementation program would be

$$\begin{aligned} &\max\{E[z_1]\} \\ &\min\{Var[z_1]\} \\ &\max\{E[z_2]\} \\ &\max\{E[z_5]\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{5}$$

The advantage of being able to use **(a)** instead of **(c)** with stochastic objectives beyond portfolio return is of course that a Var objective for each such objective can be eliminated from the equivalent deterministic implementation program. This not only simplifies data collection requirements (as it is only necessary to know the means of the relevant random variables), but also lessens the burden on computing the nondominated set.

5 Mean-Variance Nondominated Sets

We now utilize matrix notation when convenient. To prepare for the application of the four stages of the Markowitz solution procedure to multiple criteria portfolio selection problems, it is useful to study in a little greater detail the mean-variance formulation in the bottommost box of Figure 1

$$\begin{aligned} &\max\{E[z_1] = \boldsymbol{\mu}^T\mathbf{x}\} \\ &\min\{Var[z_1] = \mathbf{x}^T\boldsymbol{\Sigma}\,\mathbf{x}\,\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{6}$$

in which $\boldsymbol{\mu} \in \mathbb{R}^n$ is the expected value vector of the r_i and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is the covariance matrix of the σ_{ij}. In this problem, the efficient set is a piecewise linear path in S. The nondominated set, being the set of images of all efficient points, is piecewise parabolic in $(Var[z_1], E[z_1])$ space. This means that when portrayed in $(Stdev[z_1], E[z_1])$ space, the nondominated set is piecewise hyperbolic. Although theory and computation are customarily carried out in $(Var[z_1], E[z_1])$ space, we mention $(Stdev[z_1], E[z_1])$ space as most nondominated sets are communicated to investors in this space.

When the feasible region is the $\mathbf{1}^T\mathbf{x} = 1$ hyperplane as in

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{1}^T\mathbf{x} = 1\,\} \tag{7}$$

the efficient and nondominated sets are straightforward. The efficient set is a (single) straight line in the hyperplane, bounded at one end and unbounded at the other. The nondominated set is the top half of a (single) hyperbola. And, as a consequence of the $\mathbf{1}^T\mathbf{x} = 1$ nature of S, the efficient and nondominated sets can, after taking the Lagrangian, be obtained by formula (see for instance Campbell, Lo and Mackinlay (1997)).

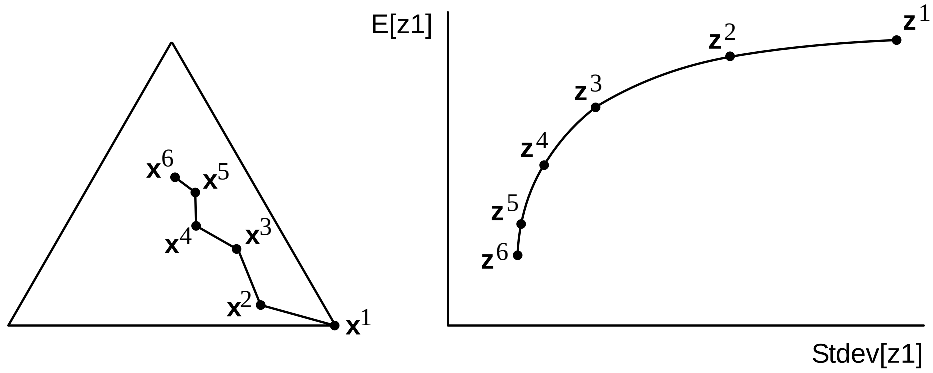

Figure 3: Piecewise linear efficient set in S (left) and piecewise hyperbolic nondominated set in $(Stdev[z_1], E[z_1])$ space (right)

However, as soon as additional constraints become involved as in

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{1}^T\mathbf{x} = 1,\ \alpha_i \leq x_i \leq \omega_i\} \tag{8}$$

thereby making S a subset of the $\mathbf{1}^T\mathbf{x} = 1$ hyperplane, the situation becomes more complicated. To illustrate, consider Figure 3 (which might correspond to a problem with about 8 securities). With a little poetic license mandated by the fact that it is not possible to draw a graph in 8-space, the graph on the left is intended to portray (a) the subset of the $\mathbf{1}^T\mathbf{x} = 1$ hyperplane that is S and (b) the efficient set which is normally a piecewise linear path. On the right in $(Stdev[z_1], E[z_1])$ space is the nondominated set (or frontier). Corresponding to the five segments of the piecewise linear path, the nondominated frontier consists of five hyperbolic segments. Note that the inverse images of the endpoints of a given nondominated hyperbolic segment are the endpoints of the efficient line segment that generates the hyperbolic segment. For instance, the inverse images of $\mathbf{z}^1$ and $\mathbf{z}^2$ are $\mathbf{x}^1$ and $\mathbf{x}^2$, respectively.

A property of a nondominated hyperbolic segment is that along the segment excluding its endpoints, the securities in a portfolio remain the same. Only their proportions change as we move along the segment. Securities can only leave a nondominated portfolio at an endpoint, and securities can only enter a nondominated portfolio if we cross over an endpoint (such as $\mathbf{z}^2$) to an adjacent nondominated hyperbolic segment. As for the number of nondominated hyperbolic segments, the larger the problem, the greater the number of nondominated hyperbolic segments. For instance, a problem with 100 securities might have 30 to 60 nondominated hyperbolic segments. Apart from when S is the entire $\mathbf{1}^T\mathbf{x} = 1$ hyperplane, mathematical programming is now the tool for obtaining information about efficient and nondominated sets.

What about software? In the past there was the IBM (1965) code. Early computer codes suffered from two problems. One was speed and the other was core (i.e., memory). Because of the amount of core required for storing a dense covariance, methods for "sparsifying" or "simplifying" the covariance matrix structure all but dominated portfolio optimization research for the next twenty years. Also, there was debate about whether a portfolio code should be "parametric" or "one-at-a-time." A *parametric* code is one that is able to define the nondominated frontier as a function of some single parameter. A *one-at-a-time* code simply computes points, one-at-a-time, on the nondominated frontier, for instance, by repetitively solving the "e-constraint" formulation[5]

$$\begin{aligned} &\min\{\, \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x}\} \\ &\text{s.t.} \quad \boldsymbol{\mu}^T\mathbf{x} \geq \rho \\ &\qquad \mathbf{x} \in S \end{aligned} \tag{9}$$

for different values of ρ. Then, with the points obtained, representations of the nondominated frontier as in Figure 4 can be prepared.

In the 1980s there was the Perold code. For achieving a breaktrough with large-scale problems (500 securities was considered large scale at the time), the code was predicated upon a covariance matrix structure sparsified according to the techniques in Markowitz and Perold (1981a and 1981b) and Perold (1984). Algorithmically drawing upon Markowitz (1956), the code was not one-at-a-time, but parametric as it was able to compute parametrically

[5]In multiple criteria optimization, programs with all objectives but one converted to constraints are often called *e-constraint* formulations.

the nondominated frontier. Having been programmed on older platforms, neither the IBM code nor Perold's code is any longer in distribution. A paper describing latest developments in portfolio optimization up until the early mid 90s is by Pardalos, Sandström and Zopounidis (1994)

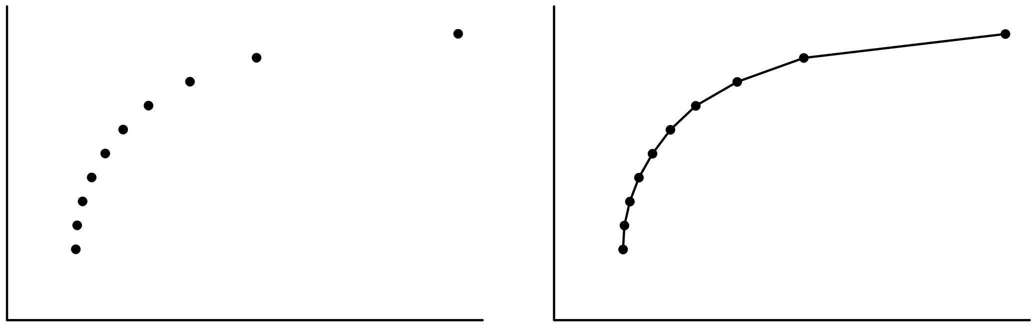

Figure 4: Dotted (left) and piecewise linear (right) representations of a nondominated frontier

As for currently, the situation is eclectic. One one hand, there are "proprietary systems" which are not really intended for university use. Rather, they are intended for integration into the computing systems of (mainly large) firms in the financial services industry. Depending upon the modifications necessary to fit into a client's computer system, the number of users, and the amount of training involved, such systems can easily run into the tens of thousands of dollars. Two such proprietary systems are FortMP, as alluded to in Mitra, Kyriakis, Lucas and Pirbhai (2003)), and QOS developed out of the works of Best (1996), Best and Kale (2000), and Best and Hlouskova (2005). Each with its own features, they are designed for large-scale problems and perform at high speed. However, they do not parametrically specify the nondominated frontier. Rather they compute points on the nondominated frontier thus resulting in both being classified as one-at-a-time.

As for codes more suitable for university use, there is the public domain code Optimizer given in the appendix of Markowitz and Todd (2000). Optimizer is parametric as it implements the critical line algorithm of Markowitz (1956). It is written in VBA (Visual Basic for Applications). However, as of this writing, it is limited to 248 securities.

One might think that commands for computing the linear segments of the efficient set and the hyperbolic segments of the nondominated frontier of (6) would be included in packages such as Cplex, Mathematica, Matlab, LINGO, SAS, and premium versions of Solver, but this is not the case. Other than for the simplistic case when S is the $\mathbf{1}^T\mathbf{x} = 1$ hyperplane, the best that can be done with the packages is to write routines within them to compute points on the nondominated frontier utilizing formulations such as (9), thus consigning us, with the packages, to an essentially one-at-a-time world.

6 Solving a Multiple Criteria Portfolio Selection Problem

Building upon knowledge gained in the previous section, we are now able to discuss the task of solving a multiple criteria portfolio selection problem. While the protocol of computing the nondominated set, communicating it to the investor, searching the nondominated set for a most preferred point, and then taking an inverse image of the selected point still remains intact, the first three stages present much greater difficulties. As for the equivalent deterministic implementation program of the bottommost box of Figure 2, different types of formulations may result. For purposes of discussion, we divide them into the three categories: (1) those with one quadratic and two or more linear objectives, (2) those with two or more quadratic and one or more linear objectives, and (3) those with one or more non-smooth objective functions (for instance $D_9(\mathbf{x})$, which is to minimize the number of securities, is non-smooth).

6.1 One Quadratic and Two or More Linear Objectives

Although many of the problems that can emerge in the bottommost box of Figure 2 cannot, given where we are in the development of multiple criteria portfolio selection, yet be effectively addressed, progress is being made on 1-quadratic 2-linear and 1-quadratic 3-linear problems in Hirschberger, Qi and Steuer (2007), and on this we comment. For instance, whereas the nondominated frontier of a mean-variance problem is piecewise hyperbolic in $(Stdev[z_1], E[z_1])$ space, the nondominated set (surface) of a 1-quadratic multi-linear multiple criteria portfolio selection is platelet-wise *hyperboloidic* in $(Stdev[z_1], E[z_1], E[z_2], \ldots)$ space. That is, the nondominated set is composed of patches, with each patch coming from the surface of a different hyperboloid. Also, whereas the efficient set in the mean-variance case is a path of linear line segments, the efficient set of a 1-quadratic multi-linear problem is a connected union of low-dimensional polyhedra in S.

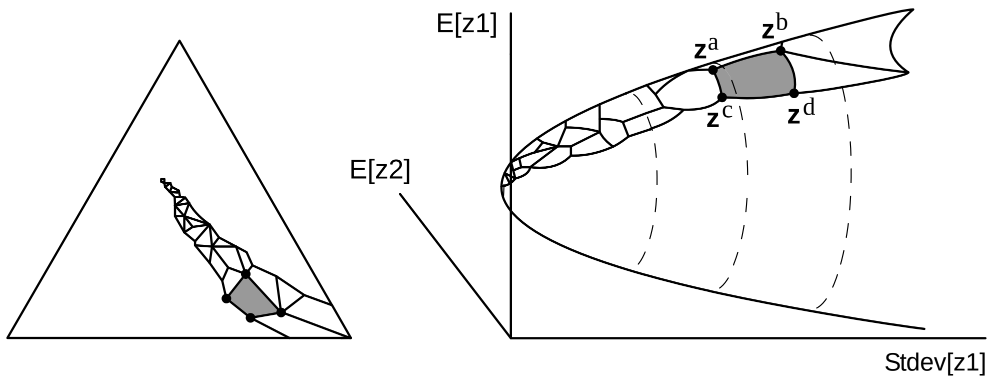

Figure 5: Efficient set in S (left) and platelet-wise hyperboloidic nondominated set of a 1-quadratic 2-linear multiple criteria portfolio selection problem in $(Stdev[z_1], E[z_1], E[z_2])$ space (right)

Consider Figure 5. On the left, the efficient set is portrayed showing the polyhedral subsets of which it is composed. On the right, the nondominated set is portrayed showing the hyperboloidic platelets of which it is composed. Note that not all platelets are of the same size and that they generally decrease in size the closer we are to the minimum standard deviation point. This is normal. Also, it is normal for not all platelets to have the same number of (platelet) *corner points*. Note the correspondence between the nondominated hyperboloidic platelets and the efficient polyhedral subsets. For instance, the platelet defined by corner points $\mathbf{z}^a$, $\mathbf{z}^b$, $\mathbf{z}^c$, $\mathbf{z}^d$ would be associated with a polyhedral subset such as the shaded one with four extreme points on the left. And as in the mean-variance case, all portfolios in the relative interior of a platelet contain the same securities, just in different proportions, and for a security to leave or for a new one to enter, one would need to cross the boundary to another platelet.

With regard to computing the nondominated set, we are able to report limited computational results using a code under development in Hirschberger, Qi and Steuer (2007). The results obtained are for 1-quadratic 2-linear problems whose covariance matrices are 100% dense. With $n = 200$ securities, 1-quadratic 2-linear problems were found to have in the neighborhood of about one thousand nondominated platelets, taking on average about ten seconds to compute. The computer used was a Dell 2.13GHz Pentium M Centrino laptop. With $n = 400$ securities, 1-quadratic 2-linear problems were found to have in the neighborhood of about two thousand nondominated platelets, taking on average about one minute to compute. These are encouraging results leading us to believe that the nondominated sets of larger problems (either in terms of number of securities or number of linear objectives) are computable in reasonable time.

Unfortunately, it is not as easy to display the nondominated set in multiple criteria portfolio selection as in mean-variance portfolio selection. In problems with criterion vectors of length three, 3D graphics can be used, but in problems with more objectives, probably about the best that can be done is to enable the investor to learn about the nondominated set while in the process of searching for a most preferred point.

As for searching the nondominated set of a problem such as in Figure 5, one approach is to discretize the nondominated set to some desired level of resolution. This can be accomplished as follows. For each polyhedral subset of the efficient set, take convex combinations of its extreme points. Because platelet size tends to increase the more distant the platelet is from the minimum standard deviation point, one would probably want to increase the number of convex combinations the further the platelet is away. Then with perhaps tens of thousands, if not hundreds of thousands, of nondominated points generated in this way, the question is how to locate a most preferred. Four strategies come to mind. One is to employ interactive multiple probing as in the Tchebycheff Method described in Steuer, Silverman and Whisman (1993). Another is to pursue a projected line search strategy as in Korhonen and Wallenius (1988) and Korhonen and Karaivanova (1999). A third is to utilize an interactive criterion vector component classification scheme as, for instance, in Miettinen (1999). And a fourth might involve the utilization of some of the visualization techniques described in Lotov, Bushenkov and Kamenev (2004).

6.2 Two or More Quadratic and One or More Linear Objectives

To illustrate what can be done in this category, consider the 2-quadratic 2-linear problem

$$\begin{aligned} &\min\{\mathbf{x}^T \boldsymbol{\Sigma}_1\, \mathbf{x}\,\} \\ &\min\{\mathbf{x}^T \boldsymbol{\Sigma}_2\, \mathbf{x}\,\} \\ &\max\{\boldsymbol{\mu}^T \mathbf{x}\} \\ &\max\{\boldsymbol{\nu}^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{10}$$

where $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are positive definite and S is defined, for instance, as in (8). Given that $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are positive definite, any convex combination

$$\bar{\boldsymbol{\Sigma}} = \lambda \boldsymbol{\Sigma}_1 + (1-\lambda)\boldsymbol{\Sigma}_2$$

(where $\lambda \in [0,1]$) renders $\bar{\boldsymbol{\Sigma}}$ positive definite. This means that the nondominated set of the 1-quadratic 2-linear

$$\begin{aligned} &\min\{\mathbf{x}^T \bar{\boldsymbol{\Sigma}}\, \mathbf{x}\,\} \\ &\max\{\boldsymbol{\mu}^T \mathbf{x}\} \\ &\max\{\boldsymbol{\nu}^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{11}$$

is a subset of the nondominated set of (10). Thus, by solving (11) for a series of different λ-values, we should be able to obtain a covering of the nondominated set of the original 2-quadratic 2-linear. The covering can then be handled in the same way as in the previous subsection.

6.3 One or More Non-Smooth Objectives

When a multiobjective equivalent deterministic problem possesses a non-smooth objective (such as to minimize the number of securities), we face major difficulties in that the problem is no longer continuous. Moreover, the nonpositive hull of the nondominated set might not even be convex. In addition, a problem might possess non-smooth constraints, for instance, in the form of semi-continuous variables (variables that are either zero or in some interval $[a, b]$ where a is materially greater than zero). Possibly, the only way to attack such problems is to utilize evolutionary algorithms such as set forth in Deb (2001).

7 Conclusions

For investors with additional concerns, portfolio selection need not only be looked at within a mean-variance framework. Steps can now be taken to integrate additional concerns into the portfolio optimization process more in accordance with their criterion status. Instead of attempting to interject additional concerns into portfolio selection by means of constraints – an *ad hoc* process that often ends prematurely because of losses in user patience – the methods that have been outlined form the basis for a new era of solution methodologies whose purposes are to converge to a final portfolio that more formally achieves optimal trade-offs among all of the criteria that the investor wishes to deem important. Of course, as with any area that is gaining momentum, more work needs to be done.

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