Reduced-Form Mortgage Valuation [1]

James B. Kau [2]
Department of Insurance, Legal Studies and Real Estate
University of Georgia

Donald C. Keenan
Department of Economics
University of Georgia

Alexey A. Smurov
Federal National Mortgage Association (Fannie Mae)

August 25, 2004

[1]We thank Garland Durham and Robert Bliss for graciously providing their computer codes. We also extend thanks to Walter J. Muller III.

[2]Corresponding author: jkau@uga.edu

Abstract

In this paper, following the recently developed reduced-form pricing literature, the authors estimate the real hazard processes of prepayment and default for a panel of conforming U.S. fixed-rate mortgages. The estimation uses a particle filter technique, and is based on an extensive set of individual mortgages, covering thirty-two consecutive years of monthly originations and twelve consecutive years of monthly observations. Having also estimated the default-free term structure, the authors then compare actual and predicted values of mortgages, in order to infer the risk-adjusted forms of the hazard processes, as well as recovery values upon default. Mortgage valuation, outside the sample used for calibration, is then performed.

GEL No. G210

James B. Kau
Department of Economics
University of Georgia
Athens, GA 30602
dkeenan@uga.edu

Donald C. Keenan
Department of Insurance,
Legal Studies and Real Estate
University of Georgia
Athens, GA 30602
jkau@uga.edu

Alexey A. Smurov
Fannie Mae
3900 Wisconsin Ave.
Washington, DC 20016
alexey_smurov@fanniemae.com

1 Introduction

Early termination, through either prepayment or default, is of particular significance in the valuation of mortgages. The possibility of such termination may be treated from either the *structural* or the *reduced-form* points of view. In the structural approach, prepayment or default is a call or put option, respectively, exercised by the borrower to minimize the cost of the contract. The decision to terminate is determined entirely within the model, and so, this structural form of modelling is often termed the *endogenous* approach. An extensive literature exists employing the structural approach in the valuation of mortgages (see, for instance, Titman and Torous (1989) or Kau, Keenan, Muller, and Epperson (1992).) While the endogenous viewpoint has yielded considerable insights into the workings of idealized mortgages, it has proven difficult to employ such models for the purpose of empirical estimation.

Just as the structural approach originated in the study of corporate debt (going at least back to the work of Merton (1974)), so has a more recent strand of asset valuation, known as the reduced-form or *exogenous* approach.[1] Compared to structural reasoning, the reduced-form point of view is a good deal less parsimonious: default or prepayment is no longer internally determined, but rather, externally imposed on the model according to some random process, so that actual termination must always ultimately come as a surprise. Borrowers may or may not then be acting rationally in the structural sense.[2] Despite its somewhat atheoretical nature, though, the reduced-form approach is entirely consistent with arbitrage-free pricing, which is, indeed, its main structural constraint. Given its flexible, open form, and its lack of dependence on unobservable factors, it is perhaps not surprising that the reduced-form approach proves much more amenable than the tighter structural form to empirical implementation.

The goal of the current paper, then, is to use the reduced-form approach to estimate default and prepayment processes, and thereby, to value mortgages. Particular attention will be paid to default, for while default is a relatively rare event, we have an extensive data set to work with, and so it proves feasible to isolate it's role. Our

[1] See the treatment in the 3rd edition of Duffie's finance text (Duffie (2001)) and the monographs by Bielecki and Rutkowski (2002) or Duffie and Singleton (2003). Seminal papers in continuous time include Madan and Unal (1998), Lando (1997), Lando (1998), and Duffie and Singleton (1999).

[2] The more basic structural models do not even permit a reduced form in the usual sense, though Duffie and Lando (2001) have shown how to formulate a structural model that does possess a well-defined reduced form.

data are the payment histories of about a million mortgages, and so, the termination processes we directly estimate will also be historical or "physical" ones. This then requires that we also estimate risk adjustments, in order to translate these real termination processes into those effectively seen by the market, and so attain the final goal of market valuation. Considerable effort is made throughout to maintain a close connection between the data, the estimation techniques, and the underlying reduced-form model, as well as to obtain all numerical estimates only from the available data: these being primarily the aforementioned mortgage histories, along with standard Treasury bond and bill series covering the observation period.

Even if one's focus is not on mortgages, we believe the present work to be an important addition to the rather sparse literature empirically testing the newly-emerged reduced-form pricing literature in finance.[3] As we argue below, our data and approach provide distinct advantages over previous empirical efforts. They allow us to estimate both additive and multiplicative risk parameters for default (and prepayment) using a common, consistent data set, and at the same time, to estimate recovery upon default, neither feature having been previously possible. They also allow us to convincingly test, under nearly ideal circumstances, the proposition that the individual, idiosyncratic components of default (or prepayment) can be completely diversified away. Finally, unlike previous literature, our results always yield stable processes, in both real and market-adjusted forms, something to which we attach considerable importance, in order that these real and market-adjusted processes be meaningfully related.

2 Data Considerations in Modelling

The mortgage data consists of the contractual specifications and payment histories of a set of fixed-rate mortgages (FRMs) originating over a thirty-two year period, between 1970 and 2001.[4] The most obvious errors or outliers having been removed, the mortgages were restricted to be thirty-year conforming loans, resulting in some 917,703 observations (see **Tables I**, **II** & **III**).[5] This body of mortgages was only

[3]We will be mostly concerned with the important works of Duffee (1999) and Driessen (2002), since they endeavor to infer the real hazard processes in a manner similar to ours. Papers such as the seminal work of Duffie and Singleton (1997), on the other hand, are only concerned with market-adjusted default, as it affects spreads.

[4]The mortgage histories were provided by Bank of America (BoA).

[5]It should be noted that a major portion of the mortgages are of rather recent vintage (see **Table III**). Our analysis would lead this to be expected, since even if

observed, however, over the period 1990 through 2001, so both left truncation and right censoring are present. In addition to ending in prepayment or default, some mortgages were sold off during this period, and so, ceased to be observed (see **Table II**). This censoring is always regarded as noninformative. The contractual details of each mortgage includes its loan size, the loan-to-value (LTV) ratio, the contract rate, and any points paid up front.[6]

Our prepayment and default data are recorded monthly. This is not inconsistent with a model that allows termination throughout time (Duffie (1998a)), but for reasons partly theoretical and partly practical, it was decided to model termination as actually only occurring at payment dates. The theoretical reason is that, in an idealized model of structural termination, borrowers will indeed only choose to default on payment dates (though the same cannot be said of prepayment). Practical concerns include not then needing to infer the unobserved pattern of termination between months.

Most reduced-form models, ours included, maintain a *doubly stochastic* character. By this, we merely mean that, not only is it uncertain whether a contract will terminate at a particular time, but that the probability by which this occurs is also uncertain beforehand. Thus, not only the actual terminations, but the probabilities of these terminations, are both modelled as stochastic processes, propagating through time. In comparison, data sets like ours are typically treated empirically using a proportional hazard framework, where all mortgages then share a common baseline hazard function[7] This specifies the shared likelihoods that the mortgages will terminate over time, though this common baseline may be adjusted up or down according to the individual characteristics (*covariates*) of the specific mortgage. As is, then, it

they had been in the same numbers originally, the mortgages of earlier origination would have be significantly diminished in number, through various forms of censoring, by the time they were observed in 1990. While the number of observations covering the end of a mortgage's life thus gets a bit thin, such behavior is of very little present value importance.

[6]Negative points represent a calculation by BoA to cover cases where, rather than pay closing costs, the borrower accepts a higher contract rate.

[7]See, for instance, Deng, Quigley, and Order (2000) in the case of mortgage default and prepayment. For work on corporate bond default and call risk, see McDonald and van de Gucht (1999). A paper that uses a proportional hazard approach to estimate a real default structure (for corporate bonds) is that of Duffie and Wang (2003). Burkhard and de Giorgi (2001) apply intensity based techniques to analyze mortgages, but use a generalized additive model (GAM) rather than a proportional hazard framework.

is difficult to square such an estimation procedure with the probability of termination being a random process, except by assuming all houses follow the same draw of that process. This would, however, vitiate the supposedly random character of the termination probabilities, since if they all come from a single draw having occurred at some remote time in the past and then repeated throughout time, the market will presumably have learned these probabilities and not consider them as random.

Our solution to the dilemma of how to treat termination probabilities as a random process, while still employing a proportional hazard framework in estimation, was to stratify the data; that is, we suppose that all mortgages in the same quarterly cohort share the same baseline hazard function, but that a different baseline hazard is drawn quarterly from the common process. While the market is assumed to know this process, there is no reason for it to know any actual draw, except in retrospect. While admittedly ad hoc, our convenient device is entirely consistent with the doubly stochastic reduced-form framework, which in its essence, only addresses a single mortgage and not the relationship among mortgages. Note that this device takes good advantage of our large data set, since there are still enough mortgages within each strata that the rare event of default can be adequately observed. While, in the reduced-form spirit, we can remain agnostic as to the precise interpretation of each termination probability's stochastic component, they may be thought to involve changes in general economic conditions, beyond those represented by the term structure, which are affecting the probability of that termination, and which are then indeed shared by all mortgages of a given cohort.[8]

3 The Model

Having considered how features of the data affect our choice of a model, we now lay out that model. There are two "latent" *baseline hazard rate processes* ${\lambda_0}^d(t)$ and ${\lambda_0}^p(t)$, representing default (d) and prepayment (p), respectively, which propagate through time according to the CIR processes with time-varying trends:

$$d\lambda_0^\ell(t) = \kappa_\ell(\theta^\ell(t) - \lambda_0^\ell(t))dt + \sigma_\ell\sqrt{\lambda_0^\ell(t)}dz_\ell^{\mathbb{P}} \qquad \ell = d, p, \tag{1}$$

[8]Under the interpretation of the stochastic component as representing external conditions, the time-varying trend of the stochastic hazard rate, as described below, will then account for patterns of termination entirely internal to a mortgage, which should then be mostly governed by the maturity of the loan, and so, similar among all mortgages. Note that both external and internal forces are allowed to interact in forming the actual hazard rate.

where the two white noise processes $z_d^{\mathbb{P}}(t)$ and $z_p^{\mathbb{P}}(t)$ are assumed to be independent.[9] The exact forms of the trends will be discussed in section **4.2**. At each payment date, these processes yield the "realized" *discrete hazard rate processes*

$$\lambda_i^\ell = e^{\boldsymbol{\beta}_\ell{}'\mathbf{x}(t(i))}\lambda_0^\ell(t(i)) \qquad \ell = d, p, \tag{2}$$

where $\mathbf{x}(t(i))$ represents the vector of covariates (discussed further in section **4.1**), and where $\boldsymbol{\beta}$ is the vector of their coefficients, to be estimated below.[10] From the present perspective, the most significant covariates are *Spread* and *Spread Squared*, since these are time varying, in that they depend on the also stochastic term structure, through the then current yield on a 10-year Treasury bond. Since termination occurs only at payment dates, the "intensity" of termination, as usually employed in reduced-form models working in continuous time, is not well defined. From a formal point of view, it is better to think, instead, in terms of the aggregated *martingale hazard process* $\Lambda_\ell^{\mathbb{P}}(t) = \sum_{t(i)\leq t} \lambda_i^\ell$.[11] It is called this since, if one considers a *stopping time* τ such that $\mathbb{P}_{\{\tau>t\}} = \prod_{t(i)\leq t}(1-\lambda_i)$ and the *jump process* $H(t) = \mathbb{1}_{\{\tau\leq t\}}$, then $\Lambda(t\wedge\tau)$ is the *compensator* of the jump process, in the sense that $H(t) - \Lambda(t\wedge\tau)$ is then a martingale (i.e., trendless), where $t\wedge\tau$ denotes $min(t,\tau)$. Schönbucher (2003) and Bélanger, Shreve, and Wong (2003) develop the theory of reduced-form default in terms of hazard processes rather than intensities, while Giesecke (2003) has introduced the terminological distinction between *intensity-based* and *compensator-based* default modelling.

We are assuming the absence of arbitrage, and therefore, that the real probability measure $\mathbb{P}$ driving our model can be transformed into an equivalent martingale measure $\mathbb{Q}$. The Girsanov theorem of Jacod and Shiryaev (1987) for random measures is general enough for our purposes and assures that the risk adjustments take the forms

$$dz_\ell^{\mathbb{Q}} = dz_\ell^{\mathbb{P}} - \nu_t^\ell dt \tag{3}$$

$$d\Lambda_\ell^{\mathbb{Q}} = \mu_t^\ell d\Lambda_\ell^{\mathbb{P}} \qquad \ell = d, p, \tag{4}$$

[9] Given how small the hazard rate can be, particularly for default, and that the observed default rate is often zero, it was judged particularly important to choose a stochastic hazard form not permitting negative values.

[10] The term $t(i)$ means the calendar time of the ith payment date.

[11] Being piecewise constant, the hazard process has no density (intensity) in the usual sense.

where the ν_t^ℓ are the usual additive drift adjustments and the μ_t^ℓ are multiplicative adjustments for jump risk. Note that, in our case, $d\Lambda^{\mathbb{P}}(t)$ signifies λ_i on payment date $t(i)$, but zero on nonpayment dates. Following convention, we take ν_t to be of the form $\nu\sqrt{\lambda_0(t)}/\sigma$, leading to the usual drift adjustment $-\nu\lambda_0(t)dt$ for $d\lambda_0(t)$.Also following convention, we take μ to simply be constant, so that $\Lambda^{\mathbb{Q}} = \Sigma_i(\mu\lambda_i) = \mu\Sigma_i\lambda_i = \mu\Lambda^{\mathbb{P}}$.

The term structure is taken to be the "extended" form Pearson and Sun (1994) of an independent two-factor time-invariant Cox, Ingersoll & Ross (CIR) model, so that in risk-neutralized form:

$$dy^\imath(t) = (\kappa_\imath(\theta^\imath - y^\imath(t)) - \nu^\imath y^\imath(t))dt + \sigma_\imath\sqrt{y^\imath(t)}dz_\imath^{\mathbb{Q}} \qquad \imath = 1, 2, \tag{5}$$

where the instantaneous interest rate is just the sum of these two factors plus a constant, so that $r(t) = y^1(t) + y^2(t) + \bar{y}$. Since the interests of this paper are not primarily in modelling interest rates, we have chosen as straightforward a term structure as possible, in the same affine class as the termination processes.[12]

Now, the time of termination is $\tau = \tau^d \wedge \tau^p$, with the associated martingale hazard process $\Lambda(t) = \Lambda_d(t) + \Lambda_p(t)) = \sum_{t(i)\le t}(\lambda_i^d + \lambda_i^p)$, in the manner of what, in the literature, is called the "first-to-default" (Duffie (1998b)). Taking this elaboration into account, the value of a mortgage $V(t(0))$ may be expressed as

$$\begin{aligned} V(t(0)) = E_{t(0)}^{\mathbb{Q}}\Big[\sum_{i=1}^{I} \mathrm{e}^{-\int_{t(0)}^{t(i)}((1-\tau_F)r(s)+\ell)ds}\Big(\prod_{j=1}^{i-1}(1-\lambda_j^d-\lambda_j^p)\Big) \\ \Big(\lambda_i^d W(i) + \lambda_i^p A(i) + (1-\lambda_i^d-\lambda_i^p)M\Big)\Big] \quad I = 360, \end{aligned} \tag{6}$$

where $W(i)$ is the *recovery value* upon default, $A(i)$ is the tax-adjusted unpaid balance due on prepayment, M is the tax-adjusted mortgage payment, indicating that the contract is to be continued, ℓ is a liquidity premium, and finally, τ_F is the effective Federal tax rate (all further described in section **4.4**).

Below, we discuss the steps of our estimation procedure in greater detail, presenting results as we go. Emphasis is primarily on estimation of the real termination processes, and secondarily on risk adjustments and valuation, with less interest shown to estimating the covariates and the term structure.

[12] While the termination processes are time-varying and the term structure processes are not, the former are only varying in the age of the mortgage; none of the processes' functional forms depend on calendar time.

4 The Estimation Procedure

4.1 Proportional Hazard Estimation of the Covariates

There is an extensive literature providing proportional hazard estimates for default and prepayment, though since such work generally stops there, the emphasis mostly is on the covariates, and not the baseline hazard. For us, it is effectively the baselines that are of interest, whereas the covariates are, to some extent, secondary. What is important is that our procedure for estimating covariates also permits nonparametric estimation of the baselines, so that no structure is being imposed on what later will be treated as random trajectories. Our technique, similar to the standard Cox partial likelihood procedure, does achieve this.

While the estimation procedure is a straightforward one, a large number of the elaborations developed for proportional hazard procedures do come into play in our estimation.[13] Not only do we introduce the critical stratification device, but we have competing risks in the form of prepayment and default, as well as time-varying covariates in the form of *Spread* and *Spread Squared*. As indicated earlier, we also have left truncation and right censoring, in addition to noninformative censoring. More significantly, because we are assuming that termination only occurs at payment dates, the actual proportional hazard procedure is a rather recently developed, non-standard one, due to Prentice and Kalbfleisch (2003). It turns out that the estimation of the covariates is done exactly the same as with the Cox partial likelihood procedure, though estimation of their covariances changes.[14]

The actual covariates used are $\mathbf{x}$ = (*Original LTV, Original LTV Squared, Contract Rate, Contract Rate Squared, Spread, Spread Squared, Log Loan Size, Log Loan Size Squared, Points, Points Squared*). *Spread* is chosen to be a two-month lag in the difference between the contract rate and the yield on a 10-year Treasury (thus involving the state of the term structure), all as a percent of the contract rate. It

[13] One feature we do not account for is individual heterogeneity, though as made clear in Ridder and Tunali (1999), stratification accounts for any group heterogeneity across quarterly cohorts. While our reasons for not treating individual heterogeneity are mainly practical (the difficulties in doing so in the present framework), we should point out that treatment of group heterogeneity, which inherently involves multiple observations, is a great deal more robust to specification error than are treatments of individual heterogeneity (see the discussion in van den Berg (2001)).

[14] See Prentice and Kalbfleisch (2003), who show that the usual asymptotic normality of the estimators continues to apply. A further modification for stratification is required in our case.

acts primarily as a normalized form of the lagged interest rate. *Spread Squared* is then defined in the obvious fashion, in order to capture nonlinear effects. *Log Loan Size* is the only variable to apparently require detrending: here, the original amount of each loan was inflated to match 2001 levels, using the Freddie Mac Home Price Index, before then being put into logs.[15] *Points* denotes the frequent practice of lenders charging an upfront fee as a percentage of the loan size. The later covariates (*Spread, Spread Squared, Log Loan Size, Log Loan Size Squared, Points, Points Squared*) are a priori thought of as being relevant to a loan with only prepayment and no possibility of default, whereas the earlier covariates (*Original LTV, Original LTV Squared, Contract Rate, Contract Rate Squared*) are a priori thought to have additional influences on FRM default. However, particularly in view of the substitution between default and prepayment emphasized in the earlier structural literature (Kau, Keenan, Muller, and Epperson (1992)), all covariates were estimated against both forms of termination. Point estimates of the covariates are reported in **Table IV**, with their standard errors below them.[16] The results appear satisfactory overall, and in general agreement with similar estimations reported in the literature.[17] All the types of covariates are significant in either their linear or their quadratic terms for both default and prepayment, except for LTV, which has no significant effect on prepayment.

In **Tables V**,**VI**, and **VII**, we report results of various specification tests of the functional forms chosen for use in **Table IV**.[18] Notably, **Table V** confirms the presence of stratification, and that, in terms of the covariates, quarterly stratifications are to be preferred to semiannual or annual ones.[19] **Table VI** indicates that, while some of the squared terms are individually insignificant, and cause some of the linear terms to be insignificant, the simple linear model is nonetheless rejected in favor of

[15]Freddie Mac's Conventional Mortgage Home Price Index (CMHPI) is available at www.freddiemac.com/finance/cmhpi.

[16]Table IV includes both the usual proportional hazard standard errors (above) and the corrected ones (below) indicated by the Prentice & Kalbfleisch estimation procedure.

[17]See, for instance, Schwartz and Torous (1989) or Deng, Quigley, and Order (2000). Schwartz & Torous used a cubic of spread in modelling prepayment, but this appears inappropriate for our data.

[18]While we are not working with the true likelihood function of the problem, Prentice and Kalbfleisch (2003) show that we have a conditional likelihood function with independent observations, so that the usual likelihood-based tests continue to apply (Wooldridge (2002), pp. 389-398).

[19]It was, a priori, judged impractical to stratify any finer than by quarters.

the quadratic version. Since covariates appear with both linear and quadratic terms, there is the possibility of non-monotonicity of their effects in the relevant range if they are oppositely-signed, or if the quadratic term is significant and its range is both positive and negative.[20] We care most, though, about the effects of LTV and contract rate on default, together with spread on prepayment, and perhaps not surprisingly, all these effects are unambiguous in their relevant ranges. *Figure 1* provides cumulative hazard rates of default differing by LTV ratio, whereas *Figure 2* shows cumulative hazard rates of prepayment varying by the initial Treasury rate. Note, in particular, that while the number of actual defaults in our sample may seem small relative to the number of mortgages ever observed (see **Table III**), the cumulative hazard rates of default, at least for mortgages of high LTV, can be far from insignificant. This potential number of defaults is, of course, being hidden by the various censoring mechanisms present, including prior prepayment, as well as by the preponderance of recent mortgages in the sample, which have not yet had much opportunity to default.

4.2 Particle Filter Estimation of the Dynamics

In order to respect the doubly stochastic nature of the termination processes, each process was separately estimated using *particle-filter* methods.[21] Since this is the heart of our estimation procedure, and particle filters are still relatively novel, we spend more time here than elsewhere describing the techniques involved.[22] The main

[20]The effect of loan size on default and the effect of contract rate on prepayment are examples of the first phenomenon, whereas the effect of points on prepayment is a not very pronounced example of the second one.

[21]In an earlier version of the paper, we used a simulated maximum likelihood estimation (SMLE) method to estimate hazards, which effectively assumes that the sampled mean rate of termination coincides with the true population mean, so that one has a trivial observation equation. This approach can still be consistent, but has poor finite sample properties. Despite the fact that there are usually many thousand mortgages under observation within any stratum, the monthly probability of termination is so low that, at least for default, the law of large numbers need not apply. As an example of the difficulties, there are numerous months without default, though the likelihood of default is presumably never actually zero.

[22]The method we use is the basic SIR particle filter of Gordon, Salmond, and Smith (1993), with stratified resampling from a "smoothed" version of the particles' empirical distribution function, as discussed in Pitt (2002). Such regularization alleviates the "sample impoverishment" problem that can otherwise plague particle filters. See Doucet, de Freitas, and Gordon (2000) for a collection of articles describing particle filter methods in detail. Our stochastic differential equations are simulated between

idea is that the latent variables propagate through continuous time according to their stochastic differential equations, which then constitute a *state model*, whereas once a month the latent variables materialize into actual defaults or prepayments, respectively, captured in *observation equations*. From the point of view of the particle filter, a stochastic differential equation just specifies a state density function

$$f_i^\ell(\lambda_{0,j}^\ell(t(i)) \mid \lambda_{0,j}^\ell(t(i-1))) \quad \ell = d, p, \tag{7}$$

yielding a discrete time transition of the realized baseline default or prepayment hazard rates for each stratum j.

The uncertainty as to which $\lambda_0^\ell(t(i))$ will arise from a given $\lambda_0^\ell(t(i-1))$ represents the one level of double stochasticity - i.e., the uncertain probability of, say, default - whereas the "noise terms" in the observation equations represent the other level of double stochasticity - i.e., given the probability of default, the uncertainty as to whether the default will actually occur. Given that houses then default, or don't, independently of one another, the number of defaults will be binomially distributed, though because the probability of either form of termination at any particular date is so low and the number of observations so high, it is appropriate to instead employ a Poisson approximation, in accordance with his celebrated limit theorem.[23] This yields the observational density functions

$$g(y_{i,j}^\ell = n \mid \lambda_{0,j}^\ell(t(i))) = \frac{(\mu_{i,j}^\ell)^n}{n!} e^{-\mu_{i,j}^\ell} \qquad \mu_{i,j}^\ell = \sum_{k=1}^{n(i,j)} \lambda_{i,k}^\ell \quad \ell = d, p, \tag{8}$$

where $y_{i,j}^\ell$ is the actual number of defaults (prepayments) in the jth stratum at the ith payment date, $\lambda_{i,k}^\ell = e^{\boldsymbol{\beta}_\ell' \mathbf{x}^k(t(i))} \lambda_{0,j}^\ell(t(i))$ is the realized probability of default (prepayment) for the kth house in the jth stratum, $\mathbf{x}^k(t(i))$ being the individual

payment dates using a standard Euler approximation technique. Pitt establishes that the particle filter method has the desired convergence property, when the number of particles grows large, as does our simulation of the stochastic differential equation, when the number of time steps grows large (see Duffie (2001)).

[23]Since there is both default and prepayment, one actually has a multinomial distribution. Furthermore, the baseline intensities must be weighted by their covariates, so the actual probabilities, while conditionally independent, are not identically distributed. Nonetheless, the resulting distribution is well approximated by two independent Poisson variables, each with a mean that is the sum of the individual means for that form of termination. See, for instance, Wang (1986) for a modern form of the appropriate Poisson approximation result. Note that the interaction of default and prepayment is second-order small, and so disappears in the limit.

covariates of that house, and $n(i,j)$ is the number of houses in the jth stratum at risk at that date.[24]

In the actual estimation, the trend term $\theta^d(t)$ was chosen to have its shape in time be an improper (i.e., non-normalized) chi-square density function, $\frac{\alpha_d}{\Gamma(\rho)2^\rho} t^{\rho-1} e^{-t/2}$, to reflect our priors, formed from earlier empirical estimations and structural-form simulations, suggesting that default should start out small, rise to a peak, and then trail off slowly. (This is also the shape indicated by the standard SDA schedule of the Public Securities Association (PSA)). In the case of prepayment, for which we have less priors, we fit a sixth-degree polynomial.

Given the parameters of the dynamic model and a prior distribution of the initial values of the state variables, $f_0^\ell(\lambda_0^\ell(t(0)); t(0))$, a particle filter simulates the likelihood that the stochastic dynamical system could have generated the known observations. It thus acts like the classical Kalman filter, but is appropriate for a nonlinear, non-Gaussian setting, such as we have, whereas a Kalman filter would, instead, yield inconsistent estimates. Roughly speaking, the particle filter works by drawing a large number of samples (particles) from the initial prior distribution of the state variables, propagating them through time, and then weighting them by the likelihood they could have resulted in the actual observation at a given payment date. At each such payment date, the more likely particles are replicated and the less likely suppressed, setting up the next month of simulation. The eventual likelihood obtained is of the form

$$\sum_{j=1}^{J} \sum_{h=1}^{R} g(y_{I,j}^\ell \mid \lambda_{0,h}^\ell(t(I))) \qquad I = 360 \qquad \ell = d, p, \tag{9}$$

where R is the number of particles per stratum, J is the number of strata (128), and $\lambda_{0,h}^\ell$ is the baseline hazard rate of the hth particle of the jth stratum.[25] This expression is then optimized to obtain the maximum likelihood estimators of the unknown parameters.[26]

---

[24]In the estimation of the termination processes, we must replace the unknown $\boldsymbol{\beta}$ with the $\hat{\boldsymbol{\beta}}$ estimated in the previous stage, but this has no effect on consistency, and given the large number of individual houses, little effect on standard errors, which remain asymptotically correct.

[25]See Pitt (2002).

[26]The maximum likelihood method is the obvious estimation technique to use throughout the paper, since typical reduced-form analysis already imposes strong assumptions on the model's distributional properties.

In **Tables VIII** & **X**, we list the parameter estimates for our two stochastic baseline equations.[27] They were obtained using 1000 particles per stratum and 30 time steps per observation period (i.e., daily steps over the thirty years). The results, particularly for default, are quite satisfactory. According to the Schwartz Information Criterion (as indicated in **Table IX**), the chi-square functional form for default is preferred to all the polynomial fits tried (going up to degree five), and more impressively, it proves superior to the encompassing class of gamma density functions (since it requires one less parameter to estimate). Turning to prepayment, in **Table X**, while the first four degrees of the polynomial fit are all significant; the fifth and sixth terms are not. This specification was nonetheless selected, since it provides the best fit among all the polynomial fits tried, up to degree seven, as indicated in **Table XI**. In addition, the sixth degree fit clearly surpasses a fit of the exponential distribution function, which serves as a smoothed version of the traditional PSA prepayment schedule.

The most striking feature in either estimation, relative to conventional PSA schedules, is the notable upturn in prepayment in the final years. The resulting shape for prepayment is very similar to that obtained by nonparametric estimation in Jegadeesh and Ju (2000), who are in accord with the usual explanation that a borrower will simply eliminate a loan whose remaining principal has become so low that the loan remains but a nuisance.[28]

In order to get a qualitative sense of the estimated baseline hazards, it may be useful to consider *Figures 3* and *4*, where the obtained trends are superimposed on clouds of stratified hazard rate estimates. A more quantitative diagnostic for judging goodness of fit starts by comparing the percent of overall actual terminations for any particular count to the percent predicted by combining all the various one-step forecasts performed by the particle filter. These results are listed in **Table XII**, where

[27] As indicated above, α_d in **Table VIII** is a multiplicative factor unnormalizing the chi-square p.d.f. The ρ in **Table VIII** serves as the single parameter of a $\chi^2(2\rho)$ p.d.f. The notation $\alpha_0,\ \alpha_1,\ \alpha_2$, etc., in **Table X** always indicates the coefficients of a polynomial, in the obvious sense.

[28] The upturn is so pronounced in the actual data that we deleted the last calendar year of observations in doing the prepayment estimation (hence the difference in the number of observations between **Tables VIII** and **X**). With a limited number of degrees of freedom allowed, the ML estimation was sacrificing the prepayment fit in the early years of a mortgage to explain the last years, taking no account of the fact that the last years are of very little present value importance. This lack of present value importance may also be why the conventional PSA schedule chooses not to reflect this phenomenon.

the possible counts have been divided into twelve cells of convenient size, for both prepayment and default. The fit for default is notably superior to that for prepayment, which is perhaps not surprising, given the focus of the paper on default and the notorious difficulties Wall Street practitioners have had in modelling prepayment. The Pearson statistics, which result from standardizing and combining the residuals of the forecasts, and which are to be compared to the degrees of freedom (12,107 and 11,976, respectively, for default and prepayment), indicate that some degree of underdispersion is present in the data relative to the model, particularly for prepayment.

The first set of standard errors listed in **Tables VIII** and **X** are the usual ones, and so, assume that all observations are independent of one another. However, when we test our residuals for serial correlation, we find indications of it among a number of the cohorts, particularly when going beyond twelve lags.[29] Our model did not provide for serial correlation; nonetheless, its presence does not affect the consistency of our parameter estimates, (Wooldridge (2002), ch.13), though it does change their standard errors. Furthermore, even within the strict setup of our model, there is no reason why there cannot be autocorrelation across cohorts, which would also affect standard errors. A unique feature of our data is that it has a natural time sequencing of the cross sections by time of origination of each stratum, and so, we are able to perform the same autoregressive tests on cross sections as we did on the time series. An issue that then arises is whether possible autocorrelation is better thought of as occurring in calendar time or mortgage time. In either case, though, results indicate that substantial autocorrelation already exists within six lags, for both default and prepayment, and so, a procedure to correct the standard errors was adopted. Since our model implies that two-lag serial correlation is induced when converting to calendar time, and since such correlation would be further aggravated by an approximation that we then need to make, it seemed advisable, instead, to perform the procedure in mortgage time.[30] We, thus, simultaneously performed Newey-West corrections, of 12

[29] We performed heteroscedasticity-robust Wald tests for the presence of pth-order autoregressive lags in each series' unstandardized residuals, obtained from the one-step ahead predictions in maturity time. We included lagged dependent variables to the same order (six and twelve) as the number of lagged residuals being tested for. The larger serial correlation at twelve lags may partly reflect the presence of unaccounted for seasonality factors. We tentatively included seasonality, but since it appeared insignificant, we eliminated it, considering it undesirable that our contract rate predictions should differ between two identical contracts with the same initial state variables, but where one originated in, say, June and one in September.

[30] The scores we have from the parameter estimation are by strata and month of maturity, and so, we no longer know in which of the three months of the stratum a

lags each, both across strata at given maturities, and across maturities for given strata (see, for instance, Conley (1999)). We include the latter correction both because of the evidence of serial correlation in maturity time, and because, in so doing, we can still account (diagonally) for the possibility that the autocorrelation across strata actually occurs, not internally in maturity time, but externally in calendar time. These adjusted standard errors are the second ones listed in Tables **VIII** and **X**.

Using our corrected parameter covariance matrices, we performed conditional moment tests for the goodness of fit of our hazard estimations, by comparing predicted and actual terminations among those sets of twelve cells introduced in **Table XII** (see Cameron and Trivedi (1998) for a discussion of the technique). The results, shown in **Table XIII**, confirm the anticipated result that we cannot, at a 5% significance level (or indeed at a 10% significance level), reject the hypothesis that our model of default is the true one, though, even at a 1% significance level, we can firmly reject the prepayment model as the true one.

4.3 Term Structure Estimation

The desire was to treat the state variables of the term structure in the same manner as the hazards, coming from monthly observations of continuously evolving state variables. Part of the reason for the choice of an (extended) independent two-factor CIR form is that it permits closed-form solutions for the yields needed in the Monte Carlo valuation procedure below, a considerable convenience. As well as having closed-form bond formulae, though, this term structure turns out to have closed-form solutions for the transition probabilities of its state variables between time periods. This then means that there was no need to engage in simulation of paths in order to perform maximum likelihood estimation of the term structure parameters.

We used two bond series, three-month bills and ten-year Treasuries, covering the same period over which our mortgages were observed, and obtained using the Bliss-Nelson-Siegel method (See Bliss (1997).) The choice of series was motivated by the desire to capture the shorter end of the term structure for discounting purposes and the longer end needed to explain termination behavior. It was decided to use 10-year bonds, rather than, say, 30-year ones, to better capture the planning horizon apparently characteristic of prepayment decisions. Given these two series, the bond formulae were inverted to yield the state values, contingent on parameter values. Since

loan originated. Thus, in exploring results in calendar time, we had to assume all the loans of a stratum originated in the middle month.

the exact likelihood that these states could have occurred is available, this likelihood can be maximized with respect to the unknown parameters to yield the estimated parameter values of the term structure.[31]

We did not put noise terms into the observation equations, but rather, treated the bond observations as exact. While the baseline hazards, on the other hand, do have noisy observation equations, the meaning there is considerably different from what noise introduced into the term structure observation equations would represent. The noise in the baseline observation equations arises from the assumed double stochasticity of the termination processes, and is then required by the model, despite the fact that it is assumed that the number of defaults and prepayments are observed without error. In contrast, any noise term introduced into the term structure observation equation would have to represent some sort of "measurement error", which useful though it might be for robustness of estimation, would be totally ad hoc, not having been built into the model, but rather, having been tacked on in the estimation stage. Thus, making our term structure estimation like our hazard estimation requires, not that there be noise in its observation equation, but that there be no noise.

One obvious difference distinguishing term structure estimation from hazard estimation is that one is now working with market data, and so, can obtain estimates already in the risk-neutral form. Further, assuming standard forms, the term structure estimation, being over time, also yields the risk-adjustment parameters needed to extract the real term structure dynamics. In contrast, because, to this point, the hazard rate dynamics and their estimation have been entirely real, we will need to use a separate procedure to obtain their risk adjustments.

The estimates of the term structure are reported in **Table XV**. While not all parameters of the two CIR factors are significant at the 10% level, their combined values are, as shown in the second panel of the **Table**. It is only these risk-adjusted combinations that are used in valuation. *Figures 5* and *6* support the typical finding (Chen and Scott (2003)) that the first factor can be associated with the 10-year Treasury rate, while the second factor can be associated with the spread between the Treasury rate and the shorter three-month bill rate.

4.4 The Monte Carlo Valuation Procedure

While this is, in some sense, the final stage of our procedure, we need to discuss it now, since it is used in the calibration of the risk adjustments of the next section.

[31]See Pearson and Sun (1994) or Dai and Singleton (2000).

Backward methods are called for in endogenous models of prepayment and default, but with termination being exogenous in reduced-form models, a forward method, such as Monte Carlo valuation, is more natural, particularly given that there are four state variables in our model.[32]

Much as with the particle filter, 1000 draws of the Monte Carlo simulations were conducted, with 30 time increments per month over the 30 year life of the mortgage. Starting values of the term structure are dictated by the actual bond values at origination, while the starting value of each baseline hazard is taken to be the beginning value of its respective trend.

The purpose of the Monte Carlo procedure is, of course, to price a mortgage by evaluating expression (6). Following Duffie and Singleton (1999), a liquidity term ℓ is introduced into the discounting formula, reflecting the illiquidity of mortgage markets relative to the market for Treasuries. The formula for the mortgage payment M and the unpaid balance $A(i)$ are contractually specified and entirely deterministic.[33] The recovery value $W(i)$ depends on whether the loan is insured, which it will be if the unpaid principal is of more than 80% LTV. [34] Actual recovery will then be $W(i) = (\phi + \psi)A(i)$, where ϕ is the percent insured and ψ is the percent recovered in the absence of insurance.[35] This formulation corresponds to the case known as

[32]See Boyle, Broadie, and Glasserman (1997) for a thorough discussion of Monte Carlo methods in asset pricing.

[33]The expression for the mortgage payment with an FRM is $M' = L[\frac{c/12}{1-\frac{1}{(1+c/12)^{360}}}]$, with c being the contract rate, and L being the loan size. Interest from such loans, though, is subject to state and Federal taxes, at the rates τ_S and τ'_F, respectively. Accounting for taxation results in the net receipts being only $M = (1 - \tau'_F)(1 - \tau_S))(1 + c)U(i - 1) + (M' - (1 + c)U(i - 1))$, where $U(i) = M'[\frac{1-\frac{1}{(1+c/12)^{360-i}}}{c/12}]$ is the unpaid principal in period i, after a payment. The receipt upon prepayment is then simply $A(i) = U(i) + M$. The Federal tax rate τ_F in the discounting term of (6) is taken to be 28% in the simulations to come, while $(1 - \tau'_F)(1 - \tau_S)$ is taken to be 67.68%, which corresponds, for instance, to $\tau'_F = \tau_F$ and $\tau_S = 6\%$.

[34]We include insurance in recovery value, since it is part of the contract from the lender's point of view, and will affect the appropriate terms of the contract used below to predict termination risk adjustments.

[35]For convenience, we have assumed matters are such that, with 100% recovery, the payoff to default is indistinguishable from a prepayment. A tax adjustment may need to be introduced if losses are deductible, but we have not bothered to explicitly express this, since even if it is present, it will simply be absorbed into our estimate of the recovery rate. The expression for $W(i)$ ought to really be $min(1, \phi + \psi)A(i)$, so that above we ignored the possibility that insurance and recovery rates together

"recovery of face value" (RFV) in the literature. We are going to assume, however, that the recovery rate is $\psi = 1-\omega$, where the loss rate ω is stochastic, ex ante, though proportionate to the current loan-to-value ratio, so $\omega = k \cdot U(i)/H(0)$, with k being a random variable and $H(0)$ the original house value. If we assume that that recovery (loss) is distributed independently of the other stochastic state variables, then (see Schönbucher (2003)), the relevant expression for the expected recovery $W^e(i)$ becomes $W^e(i) = (1 + \phi - k^e \cdot U(i)/H(0))A(i)$. Note that in considering the expectation, one must be working with the risk-neutral measure, and so the risk-adjusted expected loss parameter k^e need not indicate the real expected loss.

4.5 Risk Adjustment Calibration

In contrast to the previous stages, in finding the risk-adjustment factors for our hazard processes, we do not employ statistical estimation techniques, but instead use a calibration device, as commonly employed in finance. We have spoken of our data as being real, because their primary feature is to record actual terminations, but they also provide the contract rate for each mortgage, and thereby, the data also incorporate market risk attitudes. Indeed, in order that it be arbitrage-free, the value at origination $V(t(0))$ of a mortgage should equal the amount lent L, adjusted for points δ, so that

$$V(t(0))/L - (1-\delta) = 0. \tag{10}$$

We, thus, took samples of 100 loans each, and calculated the parameter values which minimized the sum-of-squared errors $\sum_h (V_h(t(0))/L_h - (1-\delta_h))^2$ over these chosen loans.

Given tax rates, we calibrated all remaining free parameters, as shown in **Table XVII**. Because of difficulties in identifying the prepayment parameters, however, rather than calibrate all the parameters simultaneously, we used a two-stage process. The reason for difficulties in prepayment risk-parameter calibration is not hard to find: by arbitrarily increasing the additive or multiplicative, one can always induce immediate prepayment, which then automatically balances the loan, up to points. Thus, endlessly increasing the prepayment additive or multiplicative can be an attractive way for the optimizer to reduce its sum-of-squared errors. To avoid this dilemma, we first isolated the effect of prepayment by restricting ourselves to sets of loans with loan-to-value ratios of less than 50%, so that their default component became numerically trivial, and could therefore be ignored. We then further restricted

reach unity. In the estimations, we take ϕ to be 20%, at an 80% loan to value.

these loans to only be ones having points, thus making it feasible to calibrate their prepayment parameters. Having determined these, we subsequently calibrated the remaining parameters, those of greater interest to us, using more typical sets of loans with greater than 50% LTV ratios.

While our entire empirical estimation has been built on the assumption that the real hazard processes are stable over time, nothing requires this to be the case for the risk adjustments or liquidity.[36] In particular, the measured difference between the yield on GNMA pass-through securities and the conventional mortgage rate, which is easily obtained using outside data, may be thought to serve as an imperfect measure of the required liquidity premium.[37] The average of this measure over the observation period is found to be 47.8 basis points, which is indeed not far from the liquidity premium of 38.9 basis points reported in the second stage of our calibration, when using a sample of loans from over the entire observation period. However, the average monthly absolute deviation of the GNMA liquidity measure turns out to be 17 basis points, making it quite volatile. This suggests the need to recalibrate liquidity on a monthly basis, and so we include two such calibrations in **Table XVII**, for the months of January and July 2001, using only loans from those respective months. While the liquidity measure does change noticeably, the other parameters remain reasonably stable, in comparison to one another and to the calibration over the entire observation period. As then also might be expected with volatile liquidity, the RMSE (root mean squared error) of the monthly calibrations are considerably smaller than the RMSE of the calibration done over the entire observation period.[38]

Jarrow, Lando, and Yu (2003) have formally demonstrated the natural result that, with sufficient diversification, the multiplicative risk parameter for default should approach one in value. The general idea, though, that premia for idiosyncratic jump risk can be diversified away over sufficient contracts goes back at least to Merton (1976), and should apply to the risk of prepayment, as well as to default. Our setting is a particularly appropriate one in which to test the applicability of such large-number reasoning; our sample of nearly a million mortgages is but a fraction of the

[36]Note that the reported liquidities in the first stage of calibration, the ones for exceptionally low LTV loans, exceed the second-stage ones, as might be expected.

[37]The data on mortgage rates are obtained from the FRED II database, maintained by the Federal Reserve Bank of St. Louis. The pass-through rates are published monthly in the Federal Reserve Bulletin.

[38]The RMSE we obtain over the entire observation period is similar to the 1.6% RMSE obtained in Titman and Torous (1989), who were, however, working with commercial mortgages, which feature the possibility of default, but not prepayment.

entire population, and our driving assumption has been that all loans are based on common hazard processes. Furthermore, our formulation and data uniquely put us in the position of being able to determine both the additive and multiplicative risk parameters of default and prepayment, using the very same data set employed to estimate the real processes. Duffee (1999), who uses only market data, is unable to determine default multiplicatives at all, whereas Driessen (2002) does calibrate the default multiplicatives, but only after introducing a second set of data, which being real, is rather disparate from the market data used to obtain the other parameters of the hazard processes.

As the results in **Table XVII** reveal, the multiplicative risk parameter on default is substantially greater than one in value, in violation of the diversification hypothesis. Driessen (2002) obtains similar results for the case of corporate bond default. Since we regard the mortgage market as the best candidate imaginable for the applicability of jump-risk diversification reasoning, this suggests that one is unlikely to find a situation where it satisfactorily applies. The multiplicative risk parameter for prepayment, on the other hand, is much closer to one, but still substantially greater in value.

The additive risk adjustments ν^{ℓ} that we obtain indicate aversion to both the systematic prepayment and default risk components. However, they are not so negative as to prevent the market-adjusted reversion factors from remaining mean-reverting for both the default and prepayment processes. This is in contrast to the previous results of Duffee (1999) and Driessen (2002), where in their various estimations, at least one of the default multiplicatives is always mean-averting. We would argue that mean aversion should be taken as an indication of possible misspecification. A negative market reversion factor effectively means that the market regards the hazard trend as negative. Since the initial hazard value is presumably positive, it will be far removed from this trend, and will with great likelihood explode away from it over time. It is hard to then give such processes, particularly their trends, a satisfactory interpretation, even if one regards market probabilities as merely a device for calculating present values. If, as in this previous literature, one then infers the real process from the market one, and it is stable, then the two trends, real and market, have opposite signs, and any relation between them seems somewhat obscure. Indeed, a market process being unstable would seem to call into doubt the significance of any real process that might be inferred from it. Since we have already separately estimated our real processes and do attach considerable significance to them, particularly default, it seems especially important that our market processes maintain a close connection with the real ones, and this then calls for the market-adjusted reversion factor to be positive.

It is also worthwhile emphasizing that our approach allows us to determine recovery along with the risk parameters. This is not possible in the framework of Duffee (1999) and Driessen (2002), due to the interaction of a series of restrictions in which they allow only default, use primarily market data, and take a recovery of market value (RMV) approach. Indeed, relaxing any one of these three assumptions, and we relax all three, allows one to then determine recovery. Our calibrated loss parameter indicates that, at a 100% LTV ratio, the expected loss is 28.24% of the remaining loan balance, as calibrated over the entire observation period.[39]

After calibrating our model, we performed an out-of-sample test, using a selection of mortgages coming from within our overall body of mortgages, but which do not include any of those mortgages used in the calibration. The results are reported in **Table XVIII**. Note that the loans in each sample were chosen within the indicated time period, and that the parameters used to price them are the corresponding ones for that time period, as obtained from the previous **Table**. It is difficult to make comparisons of these results with other literature, since it is uncommon for other authors to engage in out-of-sample valuation, possibly because of limited data.

4.6 Discussion

It is also of some interest to consider the impact of the various components of a mortgage: payments, prepayments, and default. Randomly picking a typical mortgage,[40] we calculate that its value would have been \$104,875 in the absence of either prepayment or default, while its value should have been \$104,251 with prepayment only, and \$100,836 with only the possibility of default.[41] On the other hand, the actual contract, with both the possibility of prepayment and default, is valued at \$102,802. Thus, prepayment, alone, lowers the value of the loan to the lender by \$624; default, alone, lowers the value by \$4,039; and both lower the value by \$2,073. The \$2,590 difference between \$2,073 and $\$624 + \$4,039 = \$4,663$ reflects the fact that prepay-

[39]While we have commented before that the loss rate involves market expectations and need not be real, it nonetheless appears quite stable and corresponds well with our priors concerning average real loss rates.

[40]The mortgage randomly picked for the example originated in July, 2001, and had an original loan balance of \$102,940, with 0.75 points, an 80% LTV, and a contract rate of 7.25%. Parameters obtained from the July 2001 sample were used in all of following pricing exercises.

[41]We continue to hold the contract rate constant at 7.25% throughout this discussion.

ment precludes default and vice versa, a phenomenon familiar from structural models of mortgage termination.[42]

Prepayment is much more common than default, but the harm of a prepayment is limited in scope, and so, default plays the greater role in explaining the spread in contract rates over Treasury rates. A default is always to a lender's disadvantage, and though rare, can lead to substantial loss, while prepayment may or may not be harmful to the lender, depending on whether interest rates are high or low. Of course, as our estimation of the spread covariates for prepayment confirms, there is significantly more prepayment in those low interest rate cases that are harmful to the lender, but nonetheless, the beneficial cases partly offset the harmful ones, and with the consequences of either being limited, prepayment requires less of a premium than does default.[43]

It is also interesting to consider the decomposition of just one mortgage. If we take our sample mortgage and decompose its predicted value, we find that the value of its likely payments is $34,295 the value of its likely prepayment settlements is $61,771, and the value of any likely default recovery is $6,737. At first, it appears a bit implausible that the value of default recovery could be anything like that of prepayment, given that the former is much more infrequent and our estimations show there to be far from full recovery. However, we are working with market-adjusted probabilities, and the aversion to default risk effectively inflates the likelihood of default, allowing for a considerable "expected" value to the default component. To see how this can be consistent with the fact that more default brings down the value of the mortgage, one has to realize that all this artificial probability being attached to default is taking away market-adjusted likelihood from prepayment and continuation, alternatives which would end with higher payments for the lender.

Our ability to predict mortgage values, while respectable, is not all that one might

[42]The fact that the combined loss in value from the opportunity to prepay and default is less than that when there is only the opportunity to default reminds us that we are not dealing with an idealized structural model. In any case, the result that the latter must weakly exceed the former holds in structural models only under the additional assumption that the lender's loss is the borrower's gain, and so there are no transactions costs.

[43]Recall that prepayment can either be to minimize the borrower's cost of the loan, or so-called "suboptimal" prepayment, occurring for a multitude of other reasons, usually associated with selling one's house. Any prepayment for the latter reason, which would not have otherwise occurred for the former reason, necessarily increases the value of the loan to the borrower.

wish. One feature undoubtedly present is variation in the personal characteristics of the borrower. An important function of lending institutions is to evaluate the creditworthiness of the borrower, and to set contract rates accordingly; however, we do not have access to this information, and so, cannot explain this part of the variation in contract rates. In building models of corporate debt, it is not unusual to introduce a separate hazard process component for every firm (see Duffee (1999) or Driessen (2002)). In our case, this would correspond to to estimating nearly a million separate stochastic processes, a clearly absurd possibility, even were there data available to support such an endeavor. Of necessity, we assume, instead, that different mortgages share common hazards, up to covariates, with these covariates including only externally observable characteristics of the mortgage.

Ultimately, we are more interested in explanation than prediction; one might have run a simple regression of our covariates against the spread of contract rates over corresponding 10-year Treasury rates, and perhaps done as good a job as our model of predicting contract rates. What our model offers, instead, is an explanation of why the resulting regression coefficients would take on the values they do, an explanation in terms of the observed behaviors of borrowers as to how they do prepay or default, and the need to explain the spread of contract rates in a way consistent with the absence of arbitrage.

5 Conclusion

Default and prepayment are uncertain events, which, if they occur, will curtail a financial contract's life. A proportional hazard model is then the obvious statistical technique by which to describe the likely survival of such contracts. However, as usually employed, the baseline of a proportional hazard model is a passive object; all the explanation lies with the covariates, which, of necessity, must be observables. Thus, in employing a proportional hazard model to describe, say, default, one is effectively assuming that the probability a contract will terminate must be commonly known, once the observable covariates become known. With a large number of similar contracts, one then knows the exact proportion that will default. Saying it another way, if the contracts are sufficiently diversifiable, the only risk premium that would be demanded for such contracts would be that which reflects the market risk of the observable covariates. In practice, the term structure is likely to be the only stochastic covariate, in which case, there would be no more sources for the risk premium on a defaultable contract than on a default-free one. Reduced-form modelling of default recognizes that there are surely elements of uncertainty contributing to default, other than those represented by the available covariates, and so, captures this by positing a "doubly stochastic" model, where, not only the act of default, but the very probability of such default, remains uncertain.

This paper unites the proportional hazard and reduced-form pricing literatures, by dividing contracts into cohorts by time of origination, and treating the baseline of each cohort as a separate draw from a common stochastic process. When applied to mortgage termination, this second source of risk then offers the possibility of explaining how the contract rate on, say, a default-free, prepayable mortgage can so significantly exceed the long-term Treasury rate. A strict proportional hazard model, with only a term structure as the time-varying covariate, could never do this without resorting to implausibly high values for the premium to the idiosyncratic risk of prepayment, a risk which should be substantially diversifiable.

In order to capture the doubly stochastic nature of the model, we use a "state-space" structure to model the baseline hazards of prepayment and of default. Each baseline model then consists of a dynamic, stochastic process, reflecting those sources of randomness common across mortgages, together with a non-dynamic "observation equation," which has the second stochastic component, reflecting the fact that the actual number of terminations is also influenced by idiosyncratic risk. We imagine the stochastic variation in the first process as reflecting the external economic conditions common to a cohort of loans, so that we also add a time-varying trend to the process. This, then, reflects the internal evolution, as they age, in the likelihood of early termination by these loans.

Kalman filtering is the traditional way to treat such a state-space model; however, this procedure provides consistent estimates only if the model is linear and Gaussian, whereas hazard models, like ours, typically are neither. We, thus, use a particle-filter approach instead, since it continues to provide consistent estimates in a non-linear, non-Gaussian setting.

Unlike a firm, a homeowner does not have a wide variety of associated contracts, individually reflecting his risk of default. On the other hand, due to the large number of loans, mortgage defaults are far more frequently observed than are firms going into default. The lack of market data for loans is compensated for by the prevalence of historical data, once one assumes that the process yielding the risk of default is the same for the various loans, up to covariates. This means, though, that we are directly estimating the actual hazards of default and prepayment, not the risk-neutral ones. To get the latter, and so be able to price mortgages, we go back to the terms of the contracts, and invoke a no-arbitrage principle: the value of the loan's payments, priced according to the market's assessment of the riskiness of the loan, must be equal to the face value of the loan. This then yields the risk adjustments that convert the actual hazard processes into risk-neutralized ones, as well as yielding expected recovery values.

Mortgages are distinguished in that they face competing risks of termination, through either prepayment of default. Except for the particular choice of functional forms for their stochastic trends, we treat these two risks in an entirely symmetric fashion. With two hazard processes and a two-state term structure, this makes for four state variables, used in the Monte Carlo estimation of mortgage values. Since the term structure also appears in the hazard covariates, we are then explicitly allowing for dependency between the stochastic interest rates and the hazards of prepayment and default.

What is absent from our model is any account of the individual characteristics of the borrower, as they might reflect the likelihood of prepayment or default. On the other hand, given its proportional hazard basis, our model would be ideally suited for including such factors; it is only that our body of data did not include these elements. A particularly interesting possibility, also limited only by the availability of data, would be to introduce information about the evolution in house prices associated with the various originating mortgages. Such a reduced-form model could then be easily compared with the earlier structural models of prepayment and default, which typically used both interest rates and house prices as their driving variables. Even as is, the current model can easily be adapted to value the various instruments available in the secondary mortgage market, or even applied to price mortgage insurance. In offering an empirically implementable model uniting proportional hazard estimation and reduced-form pricing techniques, it is hoped that this paper will encourage the application of historical data to a wide variety of those financial instruments subject to credit risk.

TABLE I

Panel A: Summary Statistics for All Loans

Variable	**Mean**	**Median**	**Std.Dev.**
Original Loan Size ($)	120,638.47	112,000.00	57,317.80
Adjusted Original Loan Size ($)	150,688.89	141,207.93	65,330.39
Original LTV (%)	78.50	80.00	17.15
Contract Rate (%)	7.62	7.50	0.80
Bond Rate at Origination (%)	6.06	5.92	0.89
Points (%)	0.02	0.00	0.80

Panel B: Summary Statistics for Defaulted Loans

Variable	**Mean**	**Median**	**Std.Dev.**
Original Loan Size ($)	96,169.93	86,750.00	47,293.58
Adjusted Original Loan Size ($)	143,786.89	128,966.10	68,775.42
Original LTV (%)	89.12	93.65	11.04
Contract Rate (%)	8.46	8.38	1.14
Bond Rate at Origination (%)	7.00	6.95	1.14
Points (%)	-0.03	0.00	0.73

Panel C: Summary Statistics for Prepaid Loans

Variable	**Mean**	**Median**	**Std.Dev.**
Original Loan Size ($)	118,285.25	112,000.00	55,196.00
Adjusted Original Loan Size ($)	159,152.39	151,517.93	64,685.40
Original LTV (%)	78.22	80.00	16.67
Contract Rate (%)	7.98	7.88	0.79
Bond Rate at Origination (%)	6.42	6.22	0.93
Points (%)	0.00	0.00	0.70

The descriptive statistics presented in Panel A were constructed using the entire sample of 917,703 loans. Those in Panels B and C use the sub-samples of 4,564 loans which defaulted during the period of 1990-2001 and the 345,997 loans prepaid during that same period, respectively. All variables were obtained from the original mortgage database, except for Bond Rate at Origination (the 10-Year Treasury Bond yield), which was computed by the Nelson-Siegel-Bliss method, using FORTRAN code provided by Robert Bliss.

TABLE II

Active Loans, Observed, Terminated, Defaulted, or Prepaid, by Year

Year	# Obs.	# Term.	% Term.	# Def.	% Def.	# Prep.	% Prep.
1990	33	0	0.00	0	0.00	0	0.00
1991	5,098	19	0.37	0	0.00	0	0.00
1992	14,870	180	1.21	0	0.00	0	0.00
1993	32,304	3,370	10.43	0	0.00	2,900	8.90
1994	34,919	2,703	7.74	7	0.02	2,426	6.94
1995	133,599	2,211	1.65	45	0.03	2,069	1.55
1996	226,158	15,742	6.96	400	0.18	14,467	6.40
1997	253,182	20,742	8.19	723	0.29	19,949	7.88
1998	319,984	63,530	19.85	767	0.24	49,915	15.60
1999	452,783	40,364	8.91	695	0.15	39,147	8.65
2000	560,304	31,950	5.70	675	0.12	30,846	5.50
2001	736,892	736,892	100.00	1,252	0.17	184,278	25.00

The Table represents the number and percentage of loans terminated, defaulted, or prepaid, relative to the number observed, for all the years under observation. Note that the sum of defaulted and prepaid loans in any year need not equal the total number of terminations, since loans can be sold, paid off, or excluded from the observation plan (as at the end of the observation period, in December 2001). Loans began being observed in August 1990.

TABLE III

Numbers of Loans Originated, Defaulted, or Prepaid, by Year of Origination

Year	# Originated	# Defaulted	% Defaulted	# Prepaid	% Prepaid
1970	240	0	0.00	201	83.75
1971	1,165	0	0.00	989	84.89
1972	1,906	0	0.00	1,526	80.06
1973	1,993	2	0.10	1,369	69.69
1974	1,097	4	0.36	704	64.18
1975	1,691	4	0.24	1,056	62.45
1976	2,514	4	0.16	1,581	62.89
1977	4,012	12	0.30	2,398	59.77
1978	3,254	14	0.43	1,983	60.94
1979	2,171	27	1.24	1,325	61.04
1980	1,096	11	1.00	628	57.30
1981	506	6	1.19	304	60.08
1982	387	6	1.55	177	45.74
1983	904	16	1.77	479	52.99
1984	838	10	1.19	486	58.00
1985	827	23	2.78	458	55.38
1986	3,748	97	2.59	2,028	54.11
1987	5,086	115	2.26	2,912	57.27
1988	2,913	92	3.16	1,797	61.69
1989	2,983	134	4.49	1,779	59.64
1990	4,642	224	4.83	2,928	63.08
1991	14,343	395	2.75	9,960	69.44
1992	44,460	690	1.55	30,488	68.42
1993	83,973	569	0.68	44,120	52.54
1994	39,859	470	1.18	23,714	59.49
1995	42,529	502	1.18	25,793	60.65
1996	23,539	189	0.80	12,280	52.17
1997	21,429	85	0.40	11,527	53.79
1998	64,837	154	0.24	21,823	33.66
1999	208,336	564	0.27	62,108	29.81
2000	146,241	143	0.10	68,899	47.11
2001	184,084	2	0.00	8,177	4.44
Total	917,703	4,564	0.50	345,997	37.70

The Table represents the number and percentage of loans that ever default or prepay, relative to the number originating, by year of origination. The total number of defaults and prepayments in the sample are also included.

TABLE IV

Stratified Proportional Hazard Estimates for Competing Risks of Default and Prepayment

Sample Size: N=917,703

	Default Model	Prepayment Model
Variable	**Estimate (Standard Error) (Corrected Standard Error)**	**Estimate (Standard Error) (Corrected Standard Error)**
Original LTV	-0.01810	-0.00032
	(0.01039)	(0.00063)
	(0.01039)	(0.00062)
Original LTV Squared	0.00055	-3.12440×10^{-6}
	(0.00007)	(4.48770×10^{-6})
	(0.00007)	(4.42600×10^{-6})
Contract Rate	0.97596	1.89780
	(0.25444)	(0.04139)
	(0.25436)	(0.04096)
Contract Rate Squared	-0.01569	-0.11102
	(0.01205)	(0.00229)
	(0.01204)	(0.00227)
Log Loan Size	-5.93580	2.80487
	(1.09216)	(0.15338)
	(1.09132)	(0.15137)
Log Loan Size Squared	0.25067	-0.09786
	(0.04676)	(0.00650)
	(0.04673)	(0.00642)
Points	-0.05412	-0.07395
	(0.02661)	(0.00265)
	(0.02659)	(0.00261)
Points Squared	0.00368	-0.01375
	(0.00297)	(0.00117)
	(0.00294)	(0.00116)
Spread	0.00794	0.04907
	(0.00762)	(0.00082)
	(0.00762)	(0.00081)
Spread Squared	0.00003	0.00008
	(0.00001)	(9.78000×10^{-7})
	(0.00001)	(9.78000×10^{-7})

Note that the Log Loan Size variable was formed by inflating a loan's size to match the Freddie Mac 2001 Conventional Mortgage Home Price Index (CMHPI) (available at `www.freddiemac.com/finance/cmhpi/`), before being taken into logs. The spread variable is the difference between the loan's contract rate and the current ten-year treasury rate, as a percent of the contract rate. It is then a time-varying covariate. The first listed standard error is the conventional one for proportional hazard models, whereas the second includes a correction required by the Prentice & Kalbfleisch modification of the Cox proportional hazard model.

TABLE V

Tests for the Presence of Quarterly Stratification

Sample Size: N=917,703

Degrees of Freedom: Q=20

Stratification	**C-Statistic**	**P-Value**
Quarterly vs. None	8804.69	0.000
Quarterly vs. Annual	1734.03	0.000
Quarterly vs. Semiannual	1297.03	0.000

The C-statistic is the clustering statistic of Ridder and Tunali (2001). Under the null hypothesis of the lower level of stratification, the statistic has a chi-squared distribution with Q degrees of freedom, where Q is the number of covariates in both models combined.

TABLE VI

Test for Joint Significance of the Squared Terms

Sample Size: N=917,703

Degrees of Freedom: Q=20

Variables	**Wald Criterion**	**P-value**
All Squared Terms	9660.43	0.000

TABLE VII

Tests for Optimal Lag Length of the Interest Rate Spread

Sample Size, N=917,703

Lag Length	**AIC**	**SBC**
0	6,384,912.8	6,385,084.5
1	6,383,579.3	6,383,751.1
2	6,381,216.9	6,381,388.5
3	6,382,805.7	6,382,977.4
4	6,385,029.5	6,385,201.2
5	6,385,041.7	6,385,213.3
6	6,385,035.4	6,385,207.2

Akaike's Information Criterion is $AIC = -2 \ln L + 2K$, whereas Schwarz's Bayesian Criterion is $SBC = -2 \ln L + \ln(N)K$, with $\ln L$ being the log-likelihood, K the number of explanatory variables, and N the number of observations. These criteria compare competing models fit to the same data, with a smaller value of the criterion being preferred.

Figure 1: **Cumulative Hazard Rate of Default by LTV ratio**

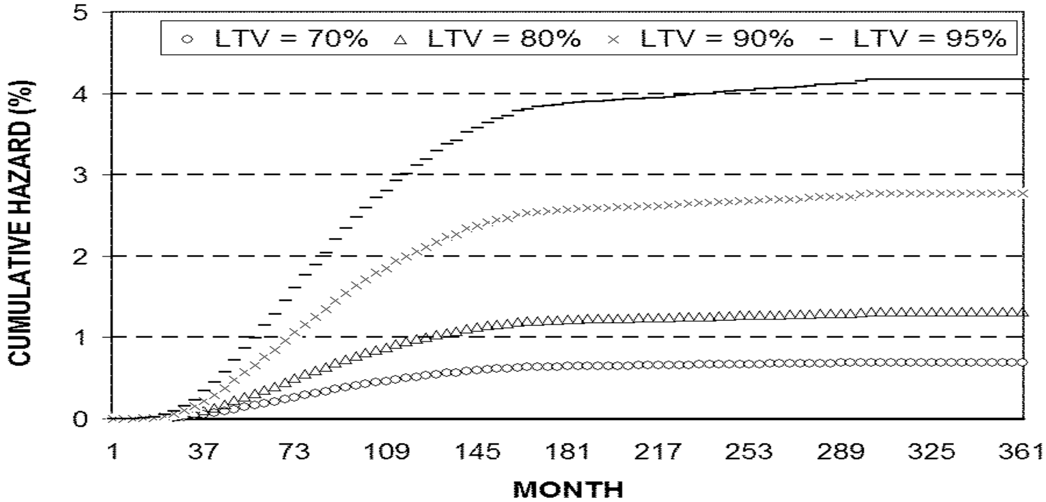
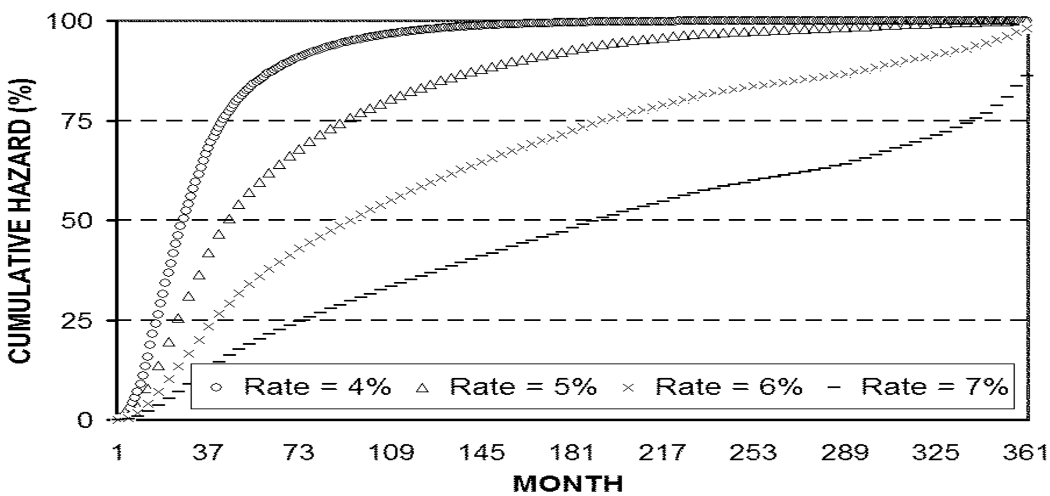


Figure 2: **Cumulative Hazard Rate of Prepayment by Interest Rate at Origination**



The predicted cumulative hazard is obtained in the usual way, as $1 - S(t(i), \hat{\beta}, \tilde{\mathbf{x}})$, where the survival function $S(t(i), \hat{\beta}, \tilde{\mathbf{x}})$ is defined by $S(t(i), \hat{\beta}, \tilde{\mathbf{x}}) = \prod_{i=1}^{t(i)} (1 - \lambda(t(i), \hat{\beta}, \tilde{\mathbf{x}}))$. All covariates in $\tilde{\mathbf{x}}$ are set to their sample means, except for ones indicated in the titles of the respective figures. In particular, the contract rate is held constant throughout.

<u>**TABLE VIII**</u>

Particle Filter Estimates for the Default Diffusion Process
Chi-Square Model

Sample Size: N=12,111

Number of Particles: R=1,000 per stratum

Propagation Method: Euler   Monthly Time Subintervals: $1/\Delta = 30$

Variable	**Estimate** **(Std. Error)** **(Corrected Std. Error)**
κ	1.382
	(0.168)
	(0.273)
α_d	38.780
	(1.839)
	(2.283)
ρ	3.587
	(0.073)
	(0.117)
σ	1.939
	(0.113)
	(0.193)
Log-Likelihood ($\ln L$)	-5,299.34

We always let κ represent the CIR reversion parameter and σ the volatility parameter. Other coefficients concern the assumed shape of the trend term $\theta_d(t)$. We thus let α_d represent a factor unnormalizing the chi-square p.d.f, while 2ρ is the single parameter characterizing a chi-square p.d.f. The first standard error of this preferred, chi-square model is the standard one; the second is a double Newey-West correction.

TABLE IX

Model Selection Tests: Default Model

Sample Size: N=12,111

Trend	**AIC**	**SBC**
3rd-Order Polynomial	10,779.8	10,824.3
4th-Order Polynomial	10,781.4	10,833.3
5th-Order Polynomial	10,784.1	10,843.3
6th-Order Polynomial	10,781.6	10,848.2
7th-Order Polynomial	10,783.6	10,857.6
Chi-Square	10,606.7	10,636.3
Gamma	10,607.9	10,644.9

Akaike's Information Criterion is $AIC = -2\ln L + 2K$, whereas Schwarz's Bayesian Criterion is $SBC = -2\ln L + \ln(N)K$, with $\ln L$ being the log-likelihood, K the number of explanatory variables, and N the number of observations. These criteria compare competing models fit to the same data, with a smaller value of the criterion being preferred.

TABLE X

Particle Filter Estimates for the Prepayment Diffusion Process
Sixth-Order Polynomial Model

Sample Size: N=11,985

Number of Particles: R=1,000 per stratum

Propagation Method: Euler Monthly Time Subintervals: $1/\Delta = 30$

Variable	**Estimate (Std. Error) (Corrected Std. Error)**
κ	0.871
	(0.057)
	(0.089)
α_0	0.388
	(0.101)
	(0.186)
α_1	1.720
	(0.213)
	(0.388)
α_2	-0.281
	(0.042)
	(0.064)
α_3	0.019
	(3.213×10^{-3})
	(3.878×10^{-3})
α_4	-6.151×10^{-4}
	(8.754×10^{-5})
	(8.970×10^{-5})
α_5	4.720×10^{-6}
	(2.876×10^{-6})
	(4.388×10^{-6})
α_6	9.354×10^{-8}
	(1.697×10^{-8})
	(1.109×10^{-7})
σ	2.297
	(0.006)
	(0.011)
Log-Likelihood ($\ln L$)	-24,832.17

We always let κ represent the CIR reversion parameter and σ the volatility parameter. Other coefficients concern the assumed shape of the trend term $\theta_p(t)$. Thus, the α_i are coefficients of polynomial terms, in the obvious sense. The first standard error of this preferred, sixth-order polynomial model is the standard one; the second is a double Newey-West correction.

Table XI

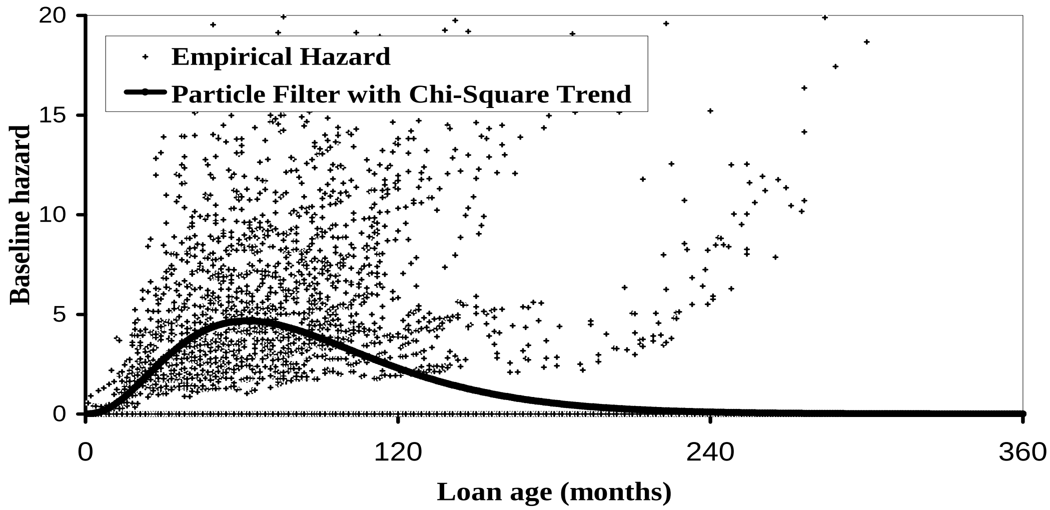
Model Selection Tests: Prepayment Model

Sample Size: N=11,985

Trend	AIC	SBC
3rd-Order Polynomial	49,777.2	49,821.5
4th-Order Polynomial	49,777.8	49,829.6
5th-Order Polynomial	49,773.5	49,832.6
6th-Order Polynomial	49,682.3	49,748.9
7th-Order Polynomial	49,682.8	49,756.7
Exponential	49,798.8	49,828.4

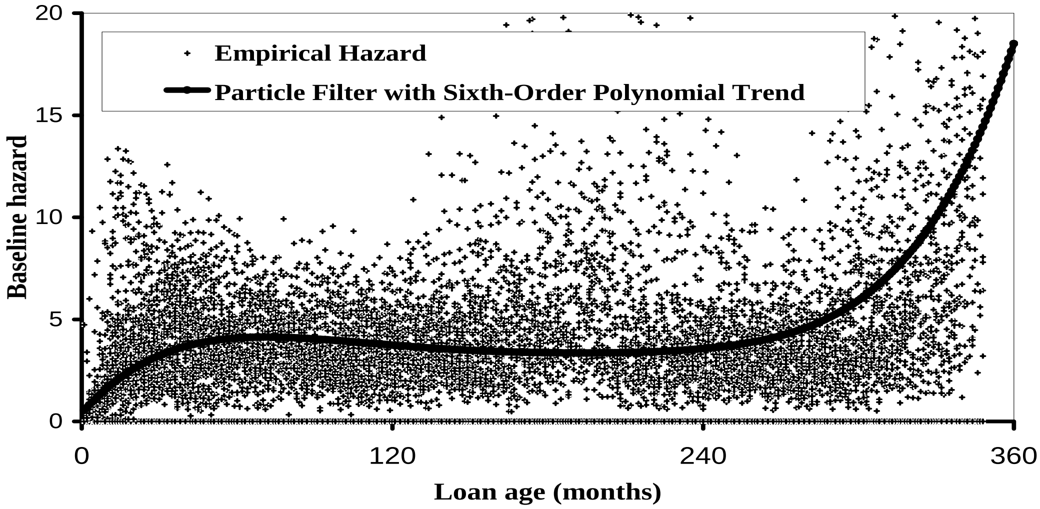
Akaike's Information Criterion is $AIC = -2\ln L + 2K$, whereas Schwarz's Bayesian Criterion is $SBC = -2\ln L + \ln(N)K$, with $\ln L$ being the log-likelihood, K the number of explanatory variables, and N the number of observations. These criteria compare competing models fit to the same data, with a smaller value of the criterion being preferred.

Figure 3: **The Trend of the Particle Filter Estimate of the Default Process vs. Stratified Empirical Proportional Hazard Estimates**



The trends of the stochastic processes for baseline default and prepayment, as specified in Section **4.2**, were evaluated using the estimated parameter values in **Table IV.** Note that many of the individual estimates above are equal to zero, especially for default towards the end of the series; indeed, there are so many there that they completely darken the abscissa axis.

Figure 4: **The Trend of the Particle Filter Estimate of the Prepayment Process vs. Stratified Empirical Proportional Hazard Estimates**



The trends of the stochastic processes for baseline default and prepayment, as specified in Section **4.2**, were evaluated using the estimated parameter values in **Table IV.** Note that many of the individual estimates above are equal to zero, especially for default towards the end of the series; indeed, there are so many there that they completely darken the abscissa axis.

TABLE XII

Predicted and Actual Probabilities

Default Model

Counts	Actual	Predicted	|Diff.|	Pearson
0	0.8264	0.8271	0.0008	0.0092
1	0.0860	0.0891	0.0031	1.3039
2	0.0421	0.0383	0.0038	4.6479
3	0.0192	0.0198	0.0006	0.2073
4	0.0118	0.0108	0.0010	1.2410
5	0.0057	0.0059	0.0002	0.1254
6	0.0031	0.0033	0.0002	0.1570
7	0.0013	0.0019	0.0006	2.3576
8	0.0014	0.0012	0.0002	0.6086
9	0.0004	0.0007	0.0003	1.6736
10	0.0006	0.0005	0.0001	0.2325
> 10	0.0019	0.0013	0.0006	3.9549

Pearson Statistic: P	10,115.53
Degrees of Freedom: $Q = N - K$	12,107.00

Prepayment Model

Counts	Actual	Predicted	|Diff.|	Pearson
0	0.3441	0.3319	0.0122	5.3556
1-10	0.4145	0.4219	0.0074	1.5568
11-20	0.0679	0.0727	0.0048	3.7343
21-30	0.0234	0.0286	0.0053	11.6077
31-40	0.0159	0.0191	0.0031	6.2293
41-50	0.0186	0.0154	0.0032	7.8740
51-60	0.0139	0.0129	0.0009	0.7869
61-70	0.0117	0.0109	0.0008	0.6217
71-80	0.0074	0.0093	0.0019	4.7016
81-90	0.0085	0.0081	0.0004	0.3008
91-100	0.0080	0.0070	0.0010	1.6987
> 100	0.0661	0.0621	0.0040	3.0733

Pearson Statistic: P	8,214.61
Degrees of Freedom: $Q = N - K$	11,976.00

The predicted probability of a termination count is its combined probability over cohorts and months, using one-step particle-filter forecasts and the now estimated parameters. This prediction is then compared to the actual percent of occasions that terminations for some cohort and month was, indeed, of that count, with the difference yielding the unstandardized residual (|Diff|). The final column calculates each category's contribution to the total Pearson statistic P, which is then compared to the degrees of freedom $Q = N - K$, with a lower value of the statistic taken to be an indication of underdispersion.

TABLE XIII

CM Tests

Degrees of Freedom: Q = 12

Dependent Variable	Test Statistic	P-Value
Default	17.32	0.138
Prepayment	43.39	0.000

For a description of the conditional moment test with count data, see Cameron and Trivedi (1998).

TABLE XIV

Summary Statistics of the Interest Rate Data

Variable	**Mean**	**Median**	**Std.Dev.**
3-month Treasury Bill	0.066	0.058	0.027
10-year Treasury Bond	0.080	0.076	0.021

TABLE XV

Exact Maximum Likelihood Estimates
of the Two-Factor Square-Root Model of the Term Structure

Observation Period: January 1970 - December 2001

Sample size: N = 384

	First Factor		**Second Factor**
Variable	**Estimate (Std. Error)**	**Estimate (Std. Error)**	**Estimate (Std. Error)**
κ	0.87		1.225
	(0.051)		(0.209)
θ	0.066		0.027
	(0.031)		(0.004)
σ	0.053		0.148
	(0.006)		(0.006)
ν	-0.073		-0.028
	(0.052)		(0.427)
$\bar{y}$		-0.038	
		(0.014)	
Log-Likelihood ($\ln L$)		3,049.16	

Estimates of the Parameter Combinations Used for Asset Pricing

$\kappa\theta$	0.006		0.033
	(0.002)		(0.004)
$\kappa + \nu$	0.015		1.197
	(0.002)		(0.403)

As always, κ is the CIR reversion coefficient for the state variable and σ is the volatility parameter, while here, θ is the constant trend, ν is the additive risk adjustment, and $\bar{y}$ is a constant, which when added to the sum of the two state variables, yields the spot rate, and makes this an "extended" two-factor CIR model. Note that the half-life, $\ln 2/\kappa$, of the first factor is found to be 7.97 years, whereas the half-life of the second is 0.56 years.

TABLE XVI

Correlation Coefficients

Variables	**Correlation**
First Factor and 10-year Treasury Bond	0.998
First Factor and 10-year Treasury Bond (First Differences)	0.989
Second Factor and Spread	-0.989
Second Factor and Spread (First Differences)	-0.998

Figure 5: **First Factor of the Term Structure Model and the 10-year Treasury Bond**

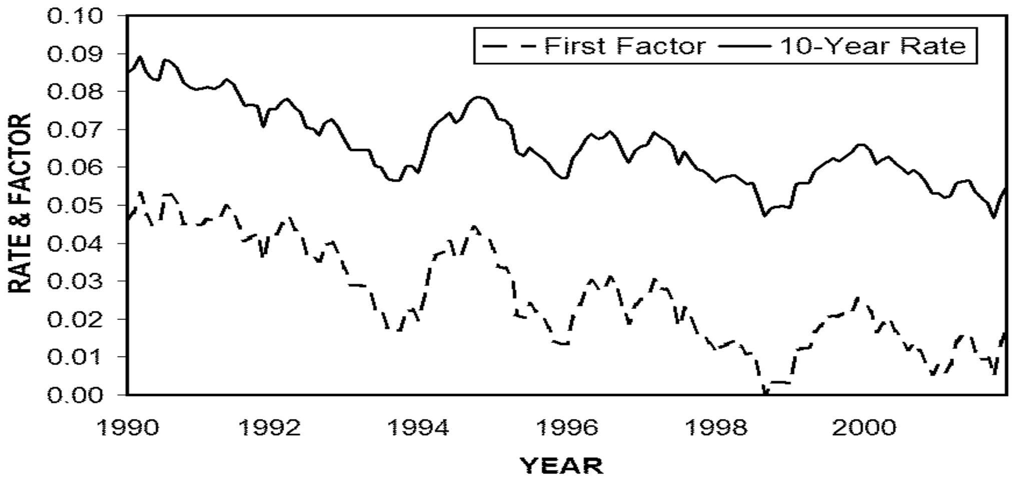


Figure 6: **Second Factor of the Term Structure Model and the Spread**

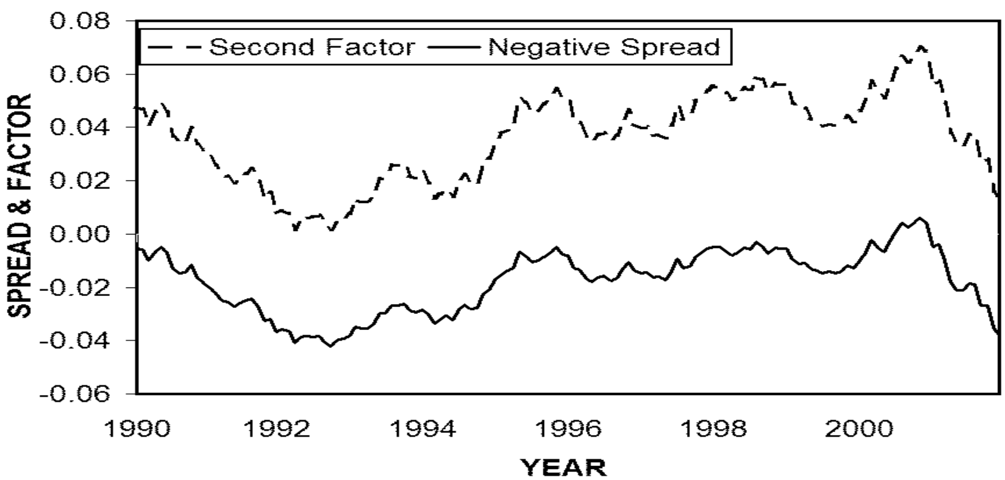


TABLE XVII

Monte Carlo Calibration Results

Number of Loans in Each Sample: H=100

Number of Simulations: N=1,000

Propagation Method: Euler

Monthly Time Subintervals: $1/\Delta = 30$

	1990-2001 Sample	**January 2001 Sample**	**July 2001 Sample**
Variable	**Estimate**	**Estimate**	**Estimate**
μ_p	1.909	1.630	1.613
ν_p	-0.323	-0.586	-0.555
Liquidity (%)[44]	0.507	0.669	0.755
RMSE (%)	1.980	1.579	1.328
μ_d	9.018	9.024	9.013
ν_d	-1.224	-1.248	-1.076
Loss Rate (%)	28.242	26.176	28.044
Liquidity (%)	0.389	0.553	0.643
RMSE (%)	1.699	1.365	1.136

[44] The liquidity premium estimated in the first stage represents only that for the loans with LTVs under 50%. It is not used in the second stage of calibration, and hence not in the subsequent tests of pricing performance.

TABLE XVIII

Out-of-Sample Pricing Errors
Number of Loans in Each Sample: H=100
Number of Simulations: N=1,000
Propagation Method: Euler
Monthly Time Subintervals: $1/\Delta = 30$

	1990-2001 Sample	**January 2001 Sample**	**July 2001 Sample**
LTV Category (%)	**RMSE (%)**	**RMSE (%)**	**RMSE (%)**
$70^{+} - 75$	2.079	1.267	1.260
$75^{+} - 80$	1.766	1.274	1.148
$80^{+} - 85$	2.131	1.473	1.296
$85^{+} - 90$	1.813	1.648	1.412
$90^{+} - 95$	1.407	1.488	1.409
$95^{+} - 100$	2.004	1.465	1.788

References

Bélanger, Alain, Steven E. Shreve, and Dennis Wong, 2003, A General Framework for Pricing Credit Risk, Working Paper.

Bielecki, Tomasz R., and Marek Rutkowski, 2002, *Credit Risk: Modeling, Valuation and Hedging.* (Springer).

Bliss, Robert R., 1997, Testing Term Structure Estimation Methods, *Advances in Futures and Options Research* 9, 191–231.

Boyle, Phelam, Mark Broadie, and Paul Glasserman, 1997, Monte Carlo methods and security pricing, *Journal of Economic Dynamics and Control* 21, 1267–1321.

Burkhard, Jürg, and Enrico de Giorgi, 2001, An Intensity Based Non-Parametric Default Model for Residential Mortgages, Working Paper, University of Zurich.

Cameron, A. Colin, and Pravin K. Trivedi, 1998, *Regression Analysis of Count Data.* (Cambridge University Press).

Chen, Ren-Raw, and Louis Scott, 2003, Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests From a Kalman Filter Model, *Journal of Real Estate Finance and Economics* 27, 143–172.

Conley, Timothy G., 1999, GMM Estimation with Cross Sectional Dependence, *Journal of Econometrics* 91, 1–45.

Dai, Qiang, and Kenneth J. Singleton, 2000, Specification Analysis of Affine Term Structure Models, *The Journal of Finance* 55, 1943–1978.

Deng, Yongheng, John M. Quigley, and Robert Van Order, 2000, Mortgage Termination, Heterogeneity and the Exercise of Mortgage Options, *Econometrica* 68, 275–307.

Doucet, Aranud, Nando de Freitas, and Neil Gordon, 2000, *Sequential Monte Carlo Methods in Practice.* (Cambridge University Press Cambridge).

Driessen, Joost, 2002, Is Default Event Risk Priced in Corporate Bonds, Working Paper, University of Amsterdam.

Duffee, Gregory, 1999, Estimating the Price of Default Risk, *Review of Financial Studies* 12, 197–226.

Duffie, Darrell, 1998a, Defaultable term Structure Models with Fractional Recovery of Par, Working Paper.

Duffie, Darrell, 1998b, First-to-Default Valuation, Working Paper, Stanford University.

Duffie, Darrell, 2001, *Dynamic Asset Pricing Theory*. (Princeton University Press Princeton New Jersey) 3rd edn.

Duffie, Darrell, and David Lando, 2001, Term structure of credit risk spreads with incomplete accounting information, *Econometrica* 69, 633–664.

Duffie, Darrell, and Kenneth J. Singleton, 1997, An Econometric Model of the Term Structure of Interest-Rate Swap Yields, *Journal of Finance* 52, 1287–1321.

Duffie, Darrell, and Kenneth J. Singleton, 1999, Modelling Term Structures of Defaultable Bonds, *Review of Financial Studies* 12, 687–720.

Duffie, Darrell, and Kenneth J. Singleton, 2003, *Credit Risk*. (Princeton University Press).

Duffie, Darrell, and Ke Wang, 2003, Multi-period corporate failure prediction with stochastic covariates, Working Paper, Stanford University.

Giesecke, Kay, 2003, Credit Risk Modeling and Valuation: An Introduction", Working Paper, Cornell University.

Gordon, Neil J., David J. Salmond, and Adrian F. M. Smith, 1993, A Novel Approach to Non-linear and Non-Gaussian Bayesian state Estimation, *IEE-Proceedings Part F* 140, 107–133.

Jacod, Jean, and Albert N. Shiryaev, 1987, *Limit Theorems for Stochastic Processes*. (Springer-Verlag).

Jarrow, Robert A., David Lando, and Fan Yu, 2003, Default Risk and Diversification: Theory and Applications, Working Paper, Cornell University.

Jegadeesh, Narashimhan, and Xiongwei Ju, 2000, A Non-Parametric Prepayment Model and Valuation of Mortgage-Backed Securities, *Journal of Fixed Income* 10, 50–67.

Kau, James B., Donald C. Keenan, Walter J. Muller, and James F. Epperson, 1992, A Generalized Valuation Model for Fixed-Rate Residential Mortgages, *Journal of Money, Credit and Banking* 24, 279–299.

Lando, David, 1997, *Mathematics of Derivative Securities*Cambridge University Press chap. Modelling Bonds and Derivatives with Default Risk, pp. 369–393.

Lando, David, 1998, On Cox Processes and Credit Risky Securities, *Review of Derivatives Research* 2, 99–120.

Madan, Dilip, and Haluk Unal, 1998, Pricing the Risk of Default, *Review of Derivatives Research* 2, 121–160.

McDonald, Cynthia G., and Linda M. van de Gucht, 1999, High-yield bond default and call risks, *Review of Economics and Statistics* 81, 409–419.

Merton, Robert C., 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *The Journal of Finance* 29, 449–470.

Merton, Robert. C., 1976, Option Pricing when Underlying Stock Returns are Discontinuous, *Journal of Financial Economics* 3, 125–144.

Pearson, Neil D., and Tong-Sheng Sun, 1994, Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll and Ross Model, *Journal of Finance* 49, 1279–1304.

Pitt, Michael K., 2002, Smooth Particle Filters for Likelihood Evaluation and Maximization, Working Paper.

Prentice, Ross L., and John D. Kalbfleisch, 2003, Mixed Discrete and Continuous Cox Regression Model, *Lifetime Data Analysis* 9, 195–210.

Ridder, Geert, and Insan Tunali, 1999, Stratified Partial Likelihood Estimation, *Journal of Econometrics* 92, 193–232.

Schönbucher, Philipp J., 2003, *Credit Derivatives Pricing Models: Model, Pricing and Implementation.* (John Wiley and Sons).

Schwartz, Eduardo S., and Walter N. Torous, 1989, Prepayment and the Valuation of Mortgage-Backed Securities, *The Journal of Finance* 44, 375–392.

Titman, Sheridan, and Walter Torous, 1989, Valuing Commercial Mortgages: An Empirical Investigation of the Contingent-Claims Approach to Pricing Risky Debt, *The Journal of Finance* 44, 345–373.

van den Berg, Gerard J., 2001, *Handbook of Econometrics*Elsevier Science B.V. vol. 5 of *Handbooks in Econometrics 2* chap. Duration Models: Specification, Identification and Multiple Durations, pp. 3380–3460.

Wang, Y. H., 1986, Coupling Methods in Approximations, *Canadian Journal of Statistics* 14, 69–74.

Wooldridge, Jeffrey M., 2002, *Econometric Analysis of Cross Section and Panel Data.* (MIT Press).