Closing the Small Open Economy Model:  
A Demographic Approach  

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Abstract  
Closing the small open economy model has been a stumbling block in studying the dynamic evolution of such models. The typical procedure of equating the after-tax return on traded bonds to the rate of time preference involves imposing an arbitrary and constraining knife-edge condition. This paper replaces the infinitely-lived representative agent framework with a plausible demographic structure. This yields a well-behaved macrodynamic equilibrium without imposing any knife-edge conditions. The equilibrium dynamics generated by the Rectangular survival function, characteristic of the Samuelson-Diamond model, closely track those corresponding to an empirically-estimated survival function. However, the Blanchard survival function tracks the data poorly in terms of absolute levels, while the closeness of its relative dynamics (following a structural change) depends on the source of the structural change.  

Keywords: Demographic structure, small open economy, macrodynamic equilibrium  

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1. Introduction

The canonical model of a small open economy assumes that it faces an exogenously fixed rate of return on traded bonds. Coupled with the conventional assumption of a constant rate of time preference, together with a constant tax rate on capital income, an interior equilibrium in a stationary economy can be sustained if and only if the implied constant after-tax return on bonds equals the given rate of time preference. This introduces a “knife-edge condition”, which although invoked by much of the international macrodynamics literature based on utility-maximizing representative agents, raises several awkward issues. First, there is no compelling reason why the required equality should hold. With the return on bonds determined in the rest of the world and the tax rate presumably set by the domestic government, there is absolutely nothing to ensure that the after-tax rate of return should equal the rate of time preference, which reflects the degree of impatience of domestic consumers. Second, requiring that this equality must always hold precludes the ability to analyze the “pure” effects of specific structural changes. For example, starting from an initial interior equilibrium, an increase in the foreign interest rate must be accompanied by either an appropriately set increase in the rate of time preference or a reduction in the tax rate in order for the knife-edge condition to be sustained and a new interior equilibrium to be reached. At best, one can determine some composite effect of the rise in the foreign interest rate, with the effect depending upon whether the necessary accommodation occurs via the rate of time preference or the tax rate. Third, the introduction of the knife-edge condition seriously limits the ability of the model to generate plausible transitional dynamics. In the absence of physical capital with associated adjustment costs, the only sustainable equilibrium is for the economy always to be in steady state, determined in part by its initial stock of traded bonds. The accumulation of traded bonds violates the transversality conditions, thus precluding any current account dynamics.

These are serious drawbacks and several approaches to circumventing these unappealing consequences have been adopted. One of the earliest was to modify the assumption of a fixed rate of return on traded bonds and the tax rate. Early examples imposing the knife-edge condition include Obstfeld (1983), Brock (1988), and Sen and Turnovsky (1989). For textbook discussions of this issue see e.g. Blanchard and Fischer (1989) and Turnovsky (1997). The term “knife-edge” was originally introduced in a different context by Harrod (1939). It refers to the need to constrain some subset of independently set parameters in order for an exogenous relationship linking them to be maintained. The small open economy model has several knife-edge characteristics, including some associated with generating endogenous growth; see Turnovsky (2002).
time preference by assuming an Uzawa (1968) utility function; see e.g. Obstfeld (1981). This approach has been criticized in that to generate plausible transitional dynamics one needs to make the counter-intuitive assumption that wealthier agents are more impatient; see Blanchard and Fischer (1989). Another early approach was to assume that the purchase of traded bonds involves convex transaction costs, typically specified as being quadratic. This assumption, first introduced by Turnovsky (1985), later by Beningo (1989), and used recently by Cacciatore and Ghironi (2012), while convenient, is arbitrary. An alternative assumption, which dates back to Bardhan (1967), is to drop the assumption of a fixed given world interest rate and to assume that the borrowing or lending rate facing an individual country depends upon its net asset position in the world economy. Although in many instances this modification is entirely plausible, particularly in the case of developing economies that are accumulating debt, the country can no longer be characterized as being a pure small economy, at least insofar as the international financial market is concerned.  

While the assumptions of a fixed world interest rate and constant rate of time preference are restrictive, the adoption of the infinitely-lived representative agent framework is also problematic. Indeed, Blanchard’s (1985) celebrated overlapping generations (OLG) model does not require any knife-edge condition to hold. Instead, for an arbitrarily set rate of time preference, capital income tax, and foreign interest rate, his model yields a convergent dynamic equilibrium time path along which the small open economy gradually adjusts its holdings of traded assets. While Blanchard’s model has the virtue of transparency and tractability, this comes at the price of being based on an unrealistic demographic survival function which has the property that mortality is independent of age and therefore cannot address lifecycle issues, as Blanchard himself acknowledged.

In this paper we build on the demographic structure pioneered by Blanchard and assume a general survival function, having the property that mortality increases with age, as intuition clearly suggests and empirical evidence strongly supports. We derive the macrodynamic equilibrium of a small open economy and show that its local dynamics can be approximated by a differential equation system which involves the evolution of: (i) the per capita accumulation of traded bonds, (ii) per capita

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2 For further discussions of alternative ways to close small open economy models, see Turnovsky (2002) and Schmitt-Grohé and Uribe (2003). Some of the more recent literature avoids the knife-edge condition by introducing financial frictions of various kinds; see e.g. Christiano, Trabandt, and Walentin (2011).
consumption, (iii) human capital at birth, and (iv) the marginal propensity to consume at birth, the latter two capturing the role of the changing demographic structure as the economy develops over time. We show that the introduction of a general demographic structure eliminates the need for imposing any knife-edge condition. Not only is this important for the aggregate dynamics of a small open economy, but also with a changing demographic composition the distributional consequences of structural shocks across cohorts also assume significance.

Because of the relative complexity of the aggregate dynamic system, we conduct our analysis of specific structural changes numerically. To do so, we employ the survival function introduced by Boucekkine, de la Croix, and Licandro (2002) (BCL) as a benchmark. This function is highly tractable and tracks the empirical data on survival for most Western economies remarkably successfully. One of the reasons for its success is that it turns out to be a very good first order approximation to the Gompertz (1825) function, which demographers find to be the most accurate representation of mortality, but being a double exponential function its dynamics are computationally intractable in a general equilibrium framework.3

We compare the predictions of the BCL survival function to those of two alternative survival functions, parameterizing them so that all three yield the same life expectancy, (E), as that implied by the BCL model. The first is a “Rectangular” survival function in which agents live with certainty until age E is reached, at which time they all die. There is no uncertainty about life expectancy, so comparing with the BCL function gives some indication of the impact of uncertainty with respect to lifespan. This comparison is of some interest for at least two reasons. First, this “one hoss shay” representation of lifespan is characteristic of the early seminal work on OLG models by Samuelson (1958) and Diamond (1965). Second, there is a growing interest in the demographic literature on the “rectangularization of the survival function”.4 Indeed, for our benchmark parameterization, as well as across a wide range of parameter variations, we find that the dynamic aggregate adjustments under the BCL and Rectangular survival functions track each other remarkably closely. This suggests that the

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3 See Bruce and Turnovsky (2013b).
4 This pertains to the trend toward a more rectangular shape of the survival function due to increased survival and concentration of deaths around the mean survival age; see e.g. Nuseelder and Mackenbach (1996) and Rossi, Rousson, and Paccaud (2012).
latter, despite its simplicity, may in fact serve as a reasonable approximation to the empirical evidence on survival, insofar as its macroeconomic implications are concerned.

The other comparison is with the Blanchard survival function and here we find that the equilibria and the transitional dynamics it generates deviate significantly from either of the other two. This is a reflection of the “perpetual youth” assumption that it embodies, and its resulting convex rather than concave (with respect to the origin) survival function, which implies an unrealistic degree of uncertainty with respect to lifespan, leading to an unrealistic level of savings. Given the fact that the other two survival functions both track empirical data much more closely suggests that, despite its analytical convenience, it is desirable to move beyond the Blanchard specification of mortality.

There is a growing literature seeking to incorporate a more realistic demographic structure into macroeconomic models. Virtually all of these papers do so within the context of a closed economy model. Some studies introduce very general mortality structures to study issues pertaining to existence and uniqueness of steady-state equilibria; see e.g. Bommier and Lee (2003), d’Albis (2007), and Lau (2009). Others adopt empirically plausible mortality functions to analyze structural and demographic changes; see e.g. Boucekkine et al. (2002), Faruqee (2003), Heijdra and Romp (2008), Heijdra and Mierau (2012), and Bruce and Turnovsky (2013a, 2013b). In general, the global dynamics of an OLG model having a realistic demographic structure is represented by a high order transcendental equation and is intractable with a neoclassical production structure; see d’Albis and Augeron-Véron (2009). In response to this, Miereau and Turnovsky (2014b) propose a linear approximation, which enables them to represent the aggregate dynamics locally as indicated above.

Of the recent literature, the present paper is closest to Heijdra and Romp (2008), who also consider a small open economy. However, their focus is very different. The present concern is with the macrodynamic equilibrium and showing how the demographic structure avoids the knife-edge condition, characteristic of much of the small open economy literature. Heijdra and Romp restrict their analysis to the steady state and the distributional consequences of various fiscal policies, something that we address briefly as well. Finally, they assume that output grows exogenously with population, whereas this paper endogenizes output through the labor-leisure choice.

Following this introduction, Section 2 sets out the analytical framework, with its demographic
structure. Section 3 derives the macrodynamic equilibrium, while Section 4 discusses the steady state. Section 5 briefly revisits the knife-edge issue in the demographic economy. Section 6 describes the calibration of the model, while numerical simulations, relating to the aggregate dynamics and the distribution across cohorts are reported in Sections 7 and 8. Section 9 concludes, while detailed derivations are relegated to the Appendix.

2. Analytical Framework

In describing the model, it is important to distinguish between calendar time and an agent’s age. In general, the variable \( X(v,t) \) refers to an agent born at time \( v \), at calendar time \( t \), when he has reached age, \( (t - v) \). The partial derivative with respect to calendar time, \( t \), represented by \( \partial X / \partial t \equiv X_t(v,t) \) describes the change in \( X \) over time for a given cohort, as it ages. The partial derivative across cohorts, represented by \( \partial X / \partial v \equiv X_v(v,t) \) describes the change across cohorts at a given point in time.

The knife-edge issue that we are addressing is due to the existence of a perfect international bond market in which agents of the small economy having a fixed rate of time preference are able to borrow or lend unlimited amounts. To highlight this source of the problem we abstract from physical capital, and thus to endogenize output we assume that labor is supplied elastically. However, the nature of labor supply is irrelevant insofar as the elimination of the knife edge condition is concerned. With inelastic labor supply, the macrodynamic equilibrium remains essentially as set out in eq. (25) below, the only difference being that the time allocation to leisure remains fixed rather than varying as in eq. (20).

At the same time we should note that physical capital may be associated with its own knife edge condition. In the limiting case where all capital is freely mobile internationally, the capital stock adjusts instantaneously to steady state, ruling out any transitional dynamics.\(^5\) Several routes are available for generating the more plausible scenario of gradual adjustment. One is to introduce convex adjustment costs of the Lucas (1967) - Hayashi (1982) type; see e.g. Sen and Turnovsky (1989). An alternative is to introduce two types of capital (e.g. physical and human) with collateral constraints

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\(^5\) See e.g. Barro and Sala-i-Martin (2004) and Turnovsky (1997).
applying only to one; see Barro and Sala-i-Martin (2004). A third is to introduce nontraded capital in addition to the freely mobile traded capital; see Brock and Turnovsky (1994). But in all cases, the knife edge problem associated with traded bonds, and being addressed here, will still remain.

### 2.1 Individual household behavior

Consider an individual born at time $v$. The probability that this individual lives to become $(t-v)$ years old is described by the survival function, $S(t-v)$, where $S'(s) = dS(s)/ds < 0$ decreases with age. The corresponding mortality rate, or instantaneous probability of death, is given by

$$\mu(t-v) = -\frac{S'(t-v)}{S(t-v)} > 0$$

(1)

The probability that the individual dies before reaching age $(t-v)$ is determined by the cumulative mortality rate:

$$M(t-v) = \int_0^{t-v} \mu(\tau)d\tau$$

(2)

Combining these two relationships, the survival function and the mortality function are related by

$$S(t-v) = e^{-M(t-v)}$$

(3)

where we assume $S(0) = e^{-M(0)} = 1$, $S(D) = e^{-M(D)} = 0$, indicating that the probability of survival of a newborn is 1, while $D$ defines the maximum age that an individual can attain.\(^6\)

Specifying the mortality function as in (3), enables us to express the discounted expected lifetime utility of an individual born at time $v$, having an isoelastic function, by

$$\int_v^{v+D} \frac{1}{\gamma} \left(C(v,t)l(v,t)^\theta\right)^\gamma e^{-\rho(t-v)-M(t-v)} dt$$

(4a)

where $C(v,t)$ denotes consumption of a traded good, and $l(v,t)$ denotes leisure, at time $t$ of an individual born at time $v$, with $\theta$ parameterizing their relative weight in utility. In addition, $\gamma$, is related to the intertemporal elasticity of substitution, $\sigma$, by $\sigma = 1/(1-\gamma)$, $\rho$ is the pure rate of time

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\(^6\) We shall treat $D$ as being finite, though the extension to an infinite $D$ (as in Blanchard, 1985) is straightforward.
discount of a newborn, while \( \rho + \mu(t-v) \) is the rate of time discount at age \((t-v)\).\(^7\)

Each individual is endowed with a unit of time that he can allocate either to leisure or to supplying labor to firms at a wage rate, \( w(t) \). These individuals also own the firms, and as an owner each one receives a share of the profit, \( \Pi(t) \), which is distributed uniformly across all cohorts, with total factor income being taxed at the constant rate \( \tau \).\(^8\) In addition to choosing leisure and consuming the traded good, the agent also accumulates internationally traded bonds, subject to his instantaneous budget constraint:

\[
F(v,t) \equiv \frac{\partial F(v,t)}{\partial t} = (1-\tau)[w(t)(1-l(v,t)) + \Pi(t)] + [(1-\tau)r^* + \mu(t-v)]F(v,t) - C(v,t) - T(t) \tag{4b}
\]

where \( F(v,t) \) denotes the traded bonds held at time \( t \) by an individual born at time \( v \), \( r^* \) is the given world rate of return on traded bonds, the income of which is taxed at the constant rate \( \tau \), and \( T(t) \) are lump sum taxes paid or rebates received, assumed to be uniform across cohorts.

Individuals are born without assets, have no bequest motive, and are not permitted to have debt if they reach the maximum attainable age \( D \). These conditions at the beginning and end of life imply \( F(v,v) = 0 = F(v,v+D) \), and individuals fully annuitize their assets. The annuities are actuarially-fair life-insured financial assets that pay conditional on the survival of the individuals. Individuals receive a premium on these annuities equal to their instantaneous probability of death, \( \mu(t-v) \), and in return when an agent dies his assets flow to the insurance company. Thus the overall rate of return received by an agent on his assets is \([(1-\tau)r^* + \mu(t-v)]\).\(^9\) Alternatively, an individual may borrow. In that case he pays a premium of \( \mu(t-v) \) and if he dies his debts are cancelled.

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\(^7\) From (1)-(3) the mortality rate and therefore the discount rate increases with age if and only if \( SS' < (S')^2 \) which is certainly met if the survival function is concave.

\(^8\) An alternative assumption is to assume that the share of profit received by the agent, \( \Pi(v,t) \), is proportional to the labor he provides, namely \( \Pi(v,t) = (1-l(v,t))/(1-l(t)) \Pi(t) \), where \( l(t) \) denotes aggregate leisure. Since this assumption, while arguably more realistic, has no bearing on the main focus of this paper, namely the role of the demographic structure in closing the small open economy model, we prefer to adopt the simpler assumption.

\(^9\) The assumption of an actuarially-fair annuities market originated with Yaari (1965) and is a central element of the Blanchard (1985) model and the vast literature that it has spawned. It provides a convenient mechanism whereby the financial wealth of decedents is recycled to the survivors in the economy. In the absence of this assumption, when agents die they leave accumulated financial capital as unintended bequests. The nature of the equilibrium depends upon how these unintended bequests are reallocated across survivors; see e.g. Hansen and Imrohoroglu (2008), Heijdra and Mierau (2012), and Bruce and Turnovsky (2013b) for examples of the consequences of annuities market imperfections. Since for the rectangular survival function the timing of death is certain, there are no unintended bequests and therefore no need for the actuarially-fair annuities market.
Optimizing (4a) subject to (4b) with respect to \( C(v,t) \), \( l(v,t) \), and \( F(v,t) \) yields:

\[
\left( C(v,t)l(v,t)^\theta \right)^{\gamma-1} l(v,t)^\theta = \lambda(v,t) \tag{5a}
\]

\[
\left( C(v,t)l(v,t)^\theta \right)^{\gamma-1} \theta l(v,t)^{\theta-1} C(v,t) = \lambda(v,t)w(t)(1 - \tau_r) \tag{5b}
\]

\[
\rho - \frac{\lambda(v,t)}{\lambda(v,t)} = (1 - \tau_r)r^* \tag{5c}
\]

Equation (5a) equates the marginal utility of consumption to the agent’s shadow value of wealth, \( \lambda(v,t) \), while (5b) equates the marginal utility of leisure to the utility value of the wage income foregone. Dividing (5b) by (5a) yields

\[
\frac{C(v,t)}{l(v,t)} = \frac{1}{\theta} w(t)(1 - \tau_r) \tag{6a}
\]

which implies that agents of all ages choose the same time-varying ratio of consumption to leisure. Equation (5c) asserts that, for any agent, his rate of return on consumption must equal the after-tax rate of return on the traded bond. Solving this equation forward in time implies

\[
\lambda(v,t) = \lambda(v,v)e^{(\rho - r^*(1 - \tau_r))(t-v)} \tag{6b}
\]

from which we see that the marginal utility of wealth of all agents grows indefinitely at the common constant rate \( \rho - r^*(1 - \tau_r) \), which is independent of their date of birth or calendar time.

In the case of the infinitely-lived representative agent economy this is the source of the knife-edge problem. The need to impose the equality \( \rho = r^*(1 - \tau_r) \) in that case arises in order to ensure that the long-run equilibrium remains bounded, as is required for a non-growing economy to be sustainable over time.\(^{10}\) This implies that the marginal utility of wealth remains constant over time. Moreover, for the frequently adopted assumption that labor is supplied inelastically, the knife-edge condition is expressed in terms of the agent’s consumption growth rate and implies complete consumption smoothing. In contrast, with heterogeneous cohorts who have finite mortality, the steady growth of

\(^{10}\) In an endogenous growth context, the knife-edge condition need not apply and an equilibrium in which consumption grows indefinitely can be sustained if the country is sufficiently patient; see Turnovsky (1996).
marginal utility of each cohort is perfectly compatible with a bounded, sustainable, aggregate
equilibrium.

Substituting for \( l(v,t) \) from (6a) into (5a), differentiating with respect to \( t \), and combining with
(5c) yields the agent’s growth rate of consumption

\[
\frac{C'_v(v,t)}{C(v,t)} = \frac{(1 - \tau_r)\rho - \theta \gamma \left( \frac{\hat{w}(t)}{w(t)} \right)}{1 - \gamma(1 + \theta)} = \psi(t)
\]

(7)

As a result of the elastically supplied labor, each agent’s growth rate of consumption varies with the
growth rate of the wage rate, at a common rate across cohorts.\(^\text{11}\) Solving (7), we can obtain the agent’s
consumption level at any arbitrary point in time, \( \tau \), relative to some earlier point in time, \( t \):

\[
C(v, \tau) = C(v, t) e^{\int_{\tau}^{t} \psi(s) \, ds}
\]

(8)

The fact that \( l(v,t) \) grows with \( C(v,t) \) in accordance with (6a) raises the possibility that
eventually \( l(v,t) \) may hit the time allocation constraint of unity. However, this turns out not to be a
practical problem insofar as our simulations are concerned. For our parameterization the steady-state
growth of consumption and leisure over the life cycle, consistent with empirical evidence, is around
0.20% per year. At this growth rate, in the case of the BCL function the constraint will be reached at
an age of 165, well in excess of the maximum lifespan \( D \). For the Blanchard case it will occur at age
of around 168 and is therefore totally irrelevant.\(^\text{12}\)

To express the agent’s consumption in terms of his financial resources we integrate the budget
constraint (4b) forward from time \( t \) and impose the transversality condition \( F(v, v + D) = 0 \). This
yields the intertemporal budget constraint

\(^{11}\) If \( \theta \gamma < 0 \) so that leisure and consumption are Edgeworth substitutes (labor supply and consumption are complements),
an increase in the growth rate of wages reduces leisure and increases the consumption growth rate, and correspondingly if
\( \theta \gamma > 0 \).

\(^{12}\) The fact that leisure declines at a constant growth rate with age implies that the fraction of time allocated to labor
\((L(v,t) = 1 - l(v,t))\) declines at an increasing rate with age, thus displaying similar concavity to the cohort labor supply function estimated by Hazan (2009, fig. 5). Of course, our labor supply function represents declining labor hours as a single household ages, whereas Hazan’s labor participation longevity function represents the declining participation with age of a population of workers. Nevertheless, we find the similarities to be of some interest.
\[
F(v,t) + \int_{t}^{v+D} \left\{(1 - \tau_{v}) \left[ w(\tau) \left(1 - l(v, \tau)\right) + \Pi(\tau) \right] - T(\tau) \right\} e^{-(1-\tau_{v})\rho_{r}(\tau-t)+M(t-\tau)-M(t-v)} d\tau \\
= \int_{t}^{v+D} C(v, \tau)e^{-(1-\tau_{v})\rho_{r}(\tau-t)+M(t-\tau)-M(t-v)} d\tau
\]

(9)

which asserts that the present value of the agent’s consumption discounted also for survival equals the discounted present value of after-tax resources. Substituting (8) into (9) yields the following expression for the consumption of an agent born at time \(v\), at calendar time \(t\), \(C(v,t)\)

\[
C(v,t) = \frac{F(v,t) + H(v,t)}{\Delta(v,t)}
\]

(10a)

where

\[
H(v,t) \equiv \int_{t}^{v+D} \left\{(1 - \tau_{v}) \left[ w(\tau) \left(1 - l(v, \tau)\right) + \Pi(\tau) \right] - T(\tau) \right\} e^{-(1-\tau_{v})\rho_{r}(\tau-t)-M(t-\tau)+M(t-v)} d\tau
\]

(10b)

denotes the agent’s discounted future income (human wealth) at time \(t\), and

\[
\Delta(v,t) \equiv \int_{t}^{v+D} e^{\int_{t}^{\tau} \theta(v) d\tau - (1-\tau_{v})\rho_{r}(\tau-t) - M(t-\tau)+M(t-v)} d\tau
\]

(10c)

is the inverse of the marginal propensity to consume out of total wealth [financial wealth \(F(v,t)\) plus human wealth \(H(v,t)\)]. Setting \(t = v\), yields the corresponding quantities at birth, where with no bequests \(F(v,v) = 0\).

### 2.2 Aggregate demography

At each instant, a cohort of size \(P(v,v) = \beta P(v)\) is born, where \(P(v,v)\) is the size of the cohort, \(P(v)\) is the size of the total population at time \(v\), and \(\beta\) is the (crude) birth rate, as measured by the average number of births per population size, and taken as constant.\(^{13}\) The number of individuals of cohort \(v\) alive at time \(t\) is \(P(v,t) = \beta P(v)e^{-M(t-v)}\). Summing over all surviving cohort members, the population at time \(t\) is \(P(t) = \beta \int_{v-D}^{t} P(v)e^{-M(t-v)} dv\). Alternatively, knowing \(P(v)\), the population alive at time \(v\), the population at time \(t\) is equal to \(P(t) = P(v)e^{n(t-v)}\), where \(n\) is the population growth rate, assumed to be constant. Equating the two expressions for \(P(t)\) leads to the following

\(^{13}\) It is straightforward to modify the specification of the birth rate to allow the child-bearing population to be some subset of the overall population.
relationship that uniquely connects the mortality rate, the birth rate, and the population growth rate:

\[
\frac{1}{\beta} = \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} dv.
\]  

(11)

Under our assumptions (i) \( \beta \) and \( n \) are constants and (ii) mortality depends only on age and is independent of calendar time, we can eliminate \( t \) from (11), and rewrite it in the form:

\[
\frac{1}{\beta} = \int_{0}^{D} e^{-n(s)-M(s)} ds
\]  

(11’)

where \( s \) indexes age; (11’) is commonly referred to as the demographic steady state (Lotka, 1998, p.60). Dividing \( P(v,t) \) by \( P(t) \), the relative size of each cohort is:

\[
p(t-v) = \frac{P(v,t)}{P(t)} = \beta e^{-n(t-v)-M(t-v)} = \beta e^{-n s-M(s)}
\]  

(12)

which in the demographic steady state also depends only on age \( s \equiv t-v \), but not on calendar time \( t \).

The dynamics of (12) are given by

\[
\frac{P_i(t-v)}{P(t-v)} = -[n + \mu(t-v)]
\]  

(13)

so that the decline in the relative size of each cohort over time reflects both its mortality rate and the overall population growth rate.

To obtain aggregate per capita values, we sum across cohorts by employing the following generic aggregator function

\[
x(t) \equiv \int_{t-D}^{t} p(t-v)X(v,t)dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} X(v,t)dv
\]  

(14)

Using this function, aggregate per capita consumption is:

\[
c(t) \equiv \int_{t-D}^{t} p(t-v)C(v,t)dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} C(v,t)dv.
\]  

(15)

Taking the time derivative of this expression and using (7) and (13), the dynamics of aggregate per
capita consumption can be expressed in the form:
\[
\dot{c}(t) \equiv \beta C(t, t) + \left[\psi(t) - \mu_c(t - \nu_i) - n\right]c(t)
\]  
(16)

where we define:\textsuperscript{14}
\[
\mu_c(t - \nu_i) \equiv \frac{\int_{t-D}^{t} \mu(t-v)p(t-v)C(v, t) dv}{\int_{t-D}^{t} p(t-v)C(v, t) dv} = \frac{1}{c(t)} \int_{t-D}^{t} \mu(t-v)p(t-v)C(v, t) dv \quad \nu_i \in (t - D, t) 
\]  
(17)

From (17) we see that \( \mu_c(t - \nu_i) \) is the ratio of the consumption given up by the dying to aggregate consumption and can be interpreted as providing an estimate of average mortality over the period \((t - D, t)\), from the consumption profile across the cohorts.\textsuperscript{15} For the general mortality function, \( \mu_c(t - \nu_i) \) varies with time, although in the special case of the Blanchard (1985) survival function it is time-invariant; see (30c) below.

Applying (14) to the cohort holdings of traded bonds, the aggregate per capita holdings of traded bonds are \( f(t) \equiv \int_{t-D}^{t} p(t-v)F(v,t) dv \). Taking the time derivative and using (4b) and (13) together with the fact that \( F(t,t) = 0 \), yields
\[
\dot{f}(t) = -\int_{t-D}^{t} \left[ \mu(t-v) + n \right] p(t-v)F(v,t) dv \\
+ \int_{t-D}^{t} p(t-v) \left( (1-\tau_s) \left[ w(t) (1-l(v,t)) + \Pi(t) \right] + [(1-\tau_s)r^{*} + \mu(t-v)] F(v,t) - C(v,t) - T(t) \right) dv 
\]  
(18)

We shall assume that the domestic government maintains a balanced budget by collecting taxes from the cohorts and rebating the revenues uniformly. Aggregating across cohorts, this is expressed in per capita terms by
\[
\tau_s \left[ w(t) (1-l(t)) + \Pi(t) \right] + \tau_s r^{*} f(t) + T(t) = 0 
\]  
(19)

\textsuperscript{14} Equation (17) is a statement of the first mean value theorem of integration. We should note that the intermediate value \( v_i \in (t - D, t) \) will in general be a function of \( t \), which case \( \mu_c(t - v_i) \) should be written as \( \mu_c(t - v_i(t)) \). We refrain from representing this explicitly, so as not to clutter notation.

\textsuperscript{15} The quantity \( \left[ \mu_c(t - v_i) + n \right] c(t) - \beta C(t, t) \) reflects the reduction in aggregate per capita consumption growth, below that of each cohort due to the arrival of newborn agents with no accumulated assets and the departure due to death of agents with assets. For this reason Heijdra and van der Ploeg (2002) initially identify this as the "generational turnover term"; see also Heijdra and Romp (2008) and Mierau and Turnovsky (2014a).
where \( l(t) \), defined in accordance with (14), specifies aggregate per capita leisure and \( \Pi(t) \), per capita profit, is defined analogously.

Summing (6a) across the surviving cohorts, we immediately see that the following analogous relationship applies to the aggregates

\[
\frac{c(t)}{l(t)} = \frac{1}{\theta} w(t)(1 - \tau_c) \tag{20}
\]

### 2.3 Firms

Output is produced by a single representative firm using labor, \( L(t) \), in accordance with a Cobb-Douglas production function. We assume that the labor market clears, in which case \( L(t) = 1 - l(t) \), and we may write per capita output as

\[
y[L(t)] = A(1 - l(t))^{\alpha} \quad 0 < \alpha < 1 \tag{21}
\]

In the absence of physical capital, the firm’s optimization problem is simple; it chooses labor to maximize profit \( \Pi(t) \equiv y[L] - wL \), so that the equilibrium wage rate and profit are

\[
w(t) = \alpha A(1 - l(t))^{\alpha - 1} \tag{22a}
\]

\[
\Pi(t) = (1 - \alpha)A(1 - l(t))^\alpha \tag{22b}
\]

From (22a) we see that increasing leisure (reducing labor supply) over time is associated with a growing wage rate.

Substituting for the equilibrium wage rate and profit from (22a) and (22b), and using the government budget constraint, (19), the aggregate accumulation equation, (18), reduces to the conventional current account relationship

\[
\dot{f}(t) = A(1 - l(t))^\alpha - c(t) + (r^* - n)f(t) \tag{23}
\]

---

16 While we abstract from accumulating capital, our production function can be viewed as including a fixed factor of production, such as land or a constant stock of non-depreciating capital.

17 The firm’s total profit is given by \( \Pi(t)P(t) \equiv (y[L] - wL)P(t) \) and clearly maximizing total profit is equivalent to maximizing per capita profit.
where the first two terms on the right hand side of (23) measures the trade balance.

3. Aggregate Equilibrium

Equations (16), and (23), yield dynamic equations for $\dot{c}$ and $\dot{f}$. In the familiar infinitely-lived representative agent economy this suffices to determine an equilibrium, in which the economy is always in steady state; see Turnovsky (1997). However, with a demographic structure the changing consumption patterns across the cohorts is an intrinsic component of the dynamics and needs to be taken into account.

3.1 Infinitely-lived representative agent economy

This is characterized by setting $\beta = n, \mu_c = 0$, $C(t,t) = c(t)$, in which case (16) reduces to $\dot{c}(t) = \psi(t)c(t)$. Combining the time derivatives of (20) and (22a), and recalling the definition of $\psi(t)$ in (7), it is straightforward to show that the dynamics of per capita consumption in the infinitely-lived representative agent model is

$$\frac{\dot{c}(t)}{c(t)} = \frac{1-\alpha l}{[1-\gamma (1+\theta)](1-l) + \theta(l(1-\gamma))} \left[ (1-\tau_r) \rho^* - \rho \right]$$

in which case the knife-edge condition required to ensure boundedness implies $\dot{c}(t) = 0$. With $c(t)$ constant, the optimality condition (20) implies $l(t)$ is constant. Assuming dynamic efficiency, $\rho^* > n$, solving the accumulation equation (23), and applying the transversality condition then requires that $f(t)$ remain constant as well, so that the economy is always in steady state, ruling out any transitional dynamics; see Turnovsky (1997).

3.2 Demographic economy

With the introduction of a general demographic structure, we see from (16) that the consumption of newborns, $C(t,t)$, as well as individual consumption growth, $\psi(t)$, play a critical role in determining the dynamics of aggregate consumption, and need to be taken into account. In the Appendix we show that the core equilibrium macroeconomic dynamics in the demographic small open economy can be
expressed by the following autonomous fourth order dynamic system expressed in terms of per capita consumption, \( c(t) \), per capita holdings of foreign assets, \( f(t) \), together with human capital at birth, \( H(t) \), and the corresponding marginal propensity to consume, \( \Delta(t) \)

\[
\begin{pmatrix}
1 + \frac{\theta \gamma}{1 - \gamma(1 + \theta)} \left[ l(c)(1 - \alpha) \right] & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{\theta^2 \gamma}{1 - \gamma(1 + \theta)} \left[ l(c)(1 - \alpha) \right] \frac{H}{c} & 0 & 1 + \theta - \frac{\theta H}{\Delta} & 0 \\
-\frac{\theta \gamma}{1 - \gamma(1 + \theta)} \left[ l(c)(1 - \alpha) \right] \Delta & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\dot{c}(t) \\
\dot{f}(t) \\
\dot{H}(t) \\
\dot{\Delta}(t) \\
\end{pmatrix}
\] (25)

where \( l(c) \) is obtained by solving the equation obtained by combining (20) and (21)

\[
l = \frac{\theta c}{(1 - \tau_y)A\alpha(1 - l)^{a-1}}
\]

so that

\[
l_c = \frac{\partial l}{\partial c} = \frac{l}{c} \left[ \frac{1 - l(c)}{1 - \alpha l(c)} \right] > 0
\]

It is important to stress that the dynamics described by (25) are functions of \( \mu_c, \mu_H, \) and \( \mu_\Delta \) which are defined by the integrals (17), (A.4), and (A.7), and are therefore in general functions of time. However, as Mierau and Turnovsky (2014b) show, given the assumption of the demographic steady state, they in fact vary only slightly over time. Their contribution to the dynamics is of second order and for practical purposes these terms can be treated as constants.\(^{18}\) Moreover, being estimates of mortality rates they are uniformly small, and can be approximated by their respective steady-state

\(^{18}\) The details of this procedure are spelled out in Mierau and Turnovsky (2014b, pp. 872-874).
levels, as derived from (28) below. Accordingly, we approximate the aggregate per capita dynamics, (25), by assuming that $\mu_c$, $\mu_H$, and $\mu_{\Delta}$ remain constant at these values.

Linearizing (25) around the steady state $(\bar{c}, \bar{f}, \bar{H}, \bar{\Delta})$, yields the system (A.9), which we employ to analyze the local dynamics. We assume that foreign bonds are accumulated gradually, while per capita consumption, human capital at birth, and the marginal propensity to consume at birth can all adjust instantaneously to any exogenous shock. This system will have a unique bounded stable transitional path if and only if there is one negative and three positive eigenvalues, in which case the local stable manifold will be one-dimensional. In principle one can establish formal conditions that ensure this required configuration of eigenvalues, although in practice these conditions are uninformative.

We should also acknowledge that while we take the demographic structure to be given there is an extensive literature that endogenizes this as part of the economic decision. For a range of examples see Becker (1981), Becker and Barro (1988), and Manuelli and Seshadri (2009). While the interdependence between the economic and demographic structures is important, it is tangential to the main issue being addressed in this paper.

4. Steady State

In steady state, the distributions of consumption, foreign bond accumulation, relative cohort size, survival, and mortality depend only upon age, $u \equiv t - \nu$, and not on calendar time. With no long-run per capita growth, per capita consumption, average leisure, per capita stock of traded bonds, and the wage rate all remain constant over time. Hence, recalling (7) each cohort’s consumption grows at the constant rate $ar{\psi} \equiv \frac{r^* (1 - \tau_r) - \rho}{1 - \gamma (1 + \theta)}$ with age, so that the consumption level of an individual of age $u$ is equal to:

$$\tilde{C}(u) = \tilde{C}_0 e^{\bar{\psi} u} \quad (26)$$

---

19 In the case of the Blanchard survival function, which assumes a constant mortality rate, $\mu_b$, across cohorts, $\mu_c = \mu_H = \mu_{\Delta} = \mu_b$ are in fact constant over time. As a result $\Delta$ is constant, in which case (25) reduces to a third order dynamic system; see Blanchard (1985).
where steady-state consumption at birth, \( \tilde{C}_0 \), is given by

\[
\tilde{C}_0 = \frac{\tilde{H}}{\Delta} \tag{27a}
\]

and

\[
\tilde{H} = \int_0^D \left[ \tilde{y} + \tilde{w} \tilde{I} (1 - \tau_y) + \tau_y r^* \tilde{f} - \theta \tilde{C}_0 e^{\tilde{\varphi}_u} \right] e^{-r^*(1-\tau_y)u-M(u)} du
\tag{27b}
\]

\[
\tilde{\Delta} = \int_0^D e^{i\tilde{\varphi}-(1-\tau_y)r^*u-M(u)} du \tag{27c}
\]

\[
\tilde{c} = \beta \tilde{C}_0 \int_0^D e^{i(\tilde{\varphi}_u-M(u))} du \tag{27d}
\]

\[
\tilde{y} = A(1-\tilde{I})^\alpha \tag{27e}
\]

\[
\tilde{w} = A\alpha (1-\tilde{I})^{\alpha-1} \tag{27f}
\]

\[
\tilde{w}(1-\tau_y)\tilde{I} = \theta \tilde{c} \tag{27g}
\]

\[
\tilde{c} = \tilde{y} + (r^* - n)\tilde{f} \tag{27h}
\]

Equation (27b) is obtained by substituting the government budget constraint, (19), and (22b) into (A.1b); (27c), (27e), (27f), and (27g) follow directly from (A.1c), (23), (22a), and (20), respectively; (27d) follows from (15) together with (26). Given the steady-state growth rate across agents, \( \tilde{\varphi}_u \), together with the mortality function, \( e^{-M(u)} \), equations (27a)-(27h) determine the steady-state values of \( \tilde{C}_0, \tilde{H}, \tilde{\Delta}, \tilde{w}, \tilde{c}, \tilde{I}, \) and \( \tilde{f} \). The steady-state values corresponding to the three specific survival functions that we employ in our numerical simulations, are then obtained by substituting (30a)-(30c) in turn [see below] into (27b)-(27d) and evaluating. The resulting expressions for the three mortality functions we consider are reported in the Appendix.

Having determined these equilibrium quantities, the corresponding steady-state values of the mortality rates, \( \mu_c, \mu_H, \) and \( \mu_\Delta \) are obtained from the steady-state relationships corresponding to the dynamic equations (25) and are respectively
\[ \tilde{\mu}_c = \beta \frac{\tilde{H}}{\Delta \tilde{c}} + \tilde{\psi} - n \]  
\[ \tilde{\mu}_H = \frac{\tilde{\psi} + (1 - \tau_r) \tilde{w} \tilde{I} + \tau_r r^* \tilde{I}}{H} - \frac{\theta}{\Delta} - \theta \tilde{\psi} - (1 - \tau_r) r^* \]  
\[ \tilde{\mu}_\lambda = \frac{1}{\Delta} + \tilde{\psi} - (1 - \tau_r) r^* \]

5. The knife-edge Revisited

As noted in Section 2.1, the source of the knife-edge problem associated with assuming an infinitely-lived representative agent in a small open economy is that under these conditions the marginal utility of wealth for all agents, and thus for the whole economy, evolve in accordance with

\[ \frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - (1 - \tau_r) r^* \]

and that with $\rho, \tau_r,$ and $r^*$ all being constant, $\lambda(t)$ will be bounded if and only if $r^*(1 - \tau_r) = \rho$. This imposes severe constraints in analyzing structural changes to the economy. For example, an increase in the foreign interest $r^*$ needs to be accompanied by either an increase in $\rho$ or $\tau_r$ if boundedness is to be maintained following the change. This makes it impossible to address the “pure” effect of an increase in $r^*$, since the effect on the economy will depend upon whether the accompanying adjustment is achieved through $\rho$ or $\tau_r$. And likewise in considering the impact of changes in $\tau_r$. \(^{20}\)

To consider whether similar constraints can apply in the demographic economy we apply the aggregator (14) to $\lambda(v,t)$ to yield the aggregate per capita marginal utility of wealth

\[ \lambda(t) \equiv \int_{t-D}^{t} p(t-v) \lambda(v,t) dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} \lambda(v,t) dv \]

Taking the time derivative of this expression and using (5c), and the derivative of (3) yields

\[ \dot{\lambda}(t) = \beta \lambda(t,t) + (\rho - (1 - \tau_r) r^* - n) \lambda(t) + \beta \int_{t-D}^{t} S'(t-v) e^{-n(t-v)} \lambda(v,t) dv \]

\(^{20}\) We should acknowledge, however, that since $r^*(1 - \tau_r)$ can deviate from $\rho$ in the short run, it is possible to study the impact of temporary changes in foreign interest rates in the representative agent framework.
Integrating by parts and simplifying we obtain
\[
\frac{\dot{\lambda}(t)}{\lambda(t)} = \left( \rho - (1 - \tau_v) r^* \right) + \beta \int_{-D}^t e^{-\eta(t^*) - M(t^*)} \left( \frac{\lambda_v(t,v)}{\lambda(t)} \right) dv
\] (29)

In general it is clear that as long as \( \lambda_v / \lambda \) varies across cohorts, (29) does not imply a knife-edge condition for a bounded solution for \( \lambda(t) \) to obtain, and that is the case for any arbitrary demographic structure. However, if \( \lambda_v / \lambda \) is constant across cohorts the last term on the right hand of (29) will be a constant in which case (29) will impose a constraint on \( \rho, \tau_v \), and \( r^* \) in order for (29) to yield a bounded solution. But a constant marginal growth rate \( \lambda_v / \lambda \) across cohorts implies a degenerate demographic structure, which essentially behaves like a representative agent.

6. Numerical Simulations

To obtain further insights we supplement the formal analysis with numerical simulations of the local dynamics and the steady-state demographic equilibrium.21 As a benchmark we use the parametric survival function proposed by Boucekkine et al. (2002):

\[
S(t - v) = \frac{e^{-\mu_0(t-v)}}{\mu_0 - 1}, \quad \text{(for } 0 \leq t - v \leq D), \quad \mu_0 > 1, \mu_1 > 0, \] (30a)

where \( \mu_0 \) and \( \mu_1 \) are parameters governing “youth” and “old age” mortality, respectively. The maximum attainable age \( D \), is determined by \( S(t - v) = 0 \) and equals \( \ln \mu_0 / \mu_1 \). We estimate the two parameters, \( \mu_0 \) and \( \mu_1 \), by nonlinear least squares, using US cohort data for 2006.22 The estimates reported in Table 1 highlight how we obtain a remarkably tight fit (\( R^2 = 0.996 \)), with highly significant parameter estimates.

For comparative purposes we also employ the Rectangular survival function

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21 We have also attempted to assess the performance of the BCL and Blanchard demographic models relative to the ad hoc cost of adjustment model by comparing their respective abilities to track the observed dynamic paths of consumption and output of small European countries (e.g. Belgium, Switzerland, Netherlands) to the shocks and policy responses associated with the recent financial crisis. Unfortunately, in all cases the frameworks are too stylized to yield any useful comparison.

22 Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org or www.humanmortality.de.
characteristic of the seminal Samuelson (1958)-Diamond (1965) model, as well as the well known Blanchard (1985) survival function

\[
S^{\text{RECT}}(t-v) = \begin{cases} 
  1, & 0 \leq t-v < D \\
  0, & t-v \geq D 
\end{cases} 
\]

(30b)

These three survival functions, together with the US data for 2006, are illustrated in Figure 1.\textsuperscript{23} Since we do not consider childhood and education, we normalize the functions so that birth corresponds to age 18. As can be seen in the figure, the BCL function tracks the actual survival data for the United States closely from age 18 until around 90. Beyond that age, its concavity does not match the data particularly well. However, we do not view that as serious since only 0.7% of the US population exceed 90 and these individuals are generally retired and are relatively inactive in the economy.\textsuperscript{24} Our estimated BCL function implies a maximum attainable age of 95.06 and life expectancy at age 18 of 78.38. These are a little low, reflecting the fact that, as Fig. 2 illustrates, the function fails to capture the outliers beyond age 90. We take the population growth rate to be 1.00% which, given the survival function, implies a birth rate of 2.24%.\textsuperscript{25} This is a little high because the population growth rate also takes into account immigration.

To preserve comparability we parameterize the Rectangular and Blanchard functions so that they both imply the same life expectancy (78.38). In the case of the Rectangular function, death occurs with certainty at that same age, and comparison with the BCL function enables us to assess the impact of mild demographic uncertainty. Being extremely concave it tracks the data much better than does the Blanchard survival function, which being convex, understates the survival rate for much of the

\textsuperscript{23} An alternative approach that would avoid specifying a specific survival function would be to use observed mortality data. To follow this line we would need to conduct the analysis in discrete time, with relevant integrals being computed by substituting directly for the survival rates, from the mortality tables. While this has some appeal from a computational standpoint, we find that for our purposes introducing a specific survival function is more convenient. First, it is clearly more practical for addressing the issue of circumventing the knife-edge. Second, it is convenient for parameterizing demographic structural changes; see Boucekkine et al. (2002). Third, with the BCL function tracking the mortality data so closely, we can view it as providing a smoothed version of the raw data, thereby enabling us to enjoy the analytical advantages of continuous-time modelling. In any event, the choice raises issues regarding the tradeoffs involved in the choice between using discrete time versus continuous time in modelling macrodynamic systems.

\textsuperscript{24} With this in mind, it might be more appropriate to refer to D as the maximum attainable economic age. We should also note that the ability of the BCL data to track mortality data closely is not restricted to the US. It does just as well for the Netherlands, for example; see Heijdra and Mierau (2012).

\textsuperscript{25} UN Population Predictions 2010. Available at: www.un.org/esa/population
distribution, and thus provides a poor match.

Table 1 summarizes the key structural parameters for the baseline economy, most of which are quite standard. The productive elasticity of labor in the Cobb-Douglas production function is $\alpha = 0.65$, the world interest rate is taken to be 5%, while the two tax rates are set at 10%. With respect to preferences, we set the intertemporal elasticity of substitution to 0.4, well within the consensus range reported by Guvenen (2006), while the elasticity of leisure in utility is set at $\theta = 1.75$, consistent with the RBC literature. As noted, in general, the rate of time preference increases with age. Hence we take $\rho = 0.035$ to be the pure rate of time preference at birth, implying a discount rate of 0.0388 for the individual of average age for the BCL survival function.

One final aspect of the calibration concerns the implications for the lifetime consumption growth rate. An implication of the assumption of perfect annuities markets is that in steady-state equilibrium, consumption increases at a constant rate over agents’ lifetime, contrary to the standard life-cycle theory of consumption. In contrast, evidence from the comprehensive National Transfer Accounts study is consistent with mild steady consumption growth over the life cycle; see the studies in Lee and Mason (2011). The implied equilibrium lifetime consumption growth rate of 0.2% per annum, is generally consistent with much of this evidence; see Tung (2011).

The implied equilibrium economic variables for the three survival functions are summarized in the bottom panel of Table 1. Focusing on the BCL survival function, it implies a ratio of foreign assets to GDP of around 1.73, of the same order of magnitude of those of Hong Kong, Luxembourg, Singapore, and Switzerland, all of which are prototype small open economies, for which the present analysis is most directly applicable. In addition, it implies that the fraction of time allocated to leisure to be around 0.762, typical of the RBC literature, while the marginal propensity to consume at birth out of wealth is approximately 0.048, which is also well within the plausible range. We may also note the following dynamic characteristics of the benchmark linearized systemnamley that it has one stable eigenvalue $\lambda_1 = -0.019$ and yields the values $\mu_c = 0.0131$, $\mu_H = 0.0041$, $\mu_A = 0.0048$. Moreover, since for all the shocks considered we find $|d\mu| < 0.0006$ across steady states, this suggests

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26 These data are obtained from the IMF elibrary database, [http://www.elibrary-data.imf.org](http://www.elibrary-data.imf.org).
that on average, the rate of adjustment over time, \( d\mu_t(\tau_t - t)/dt \), will indeed be extremely small.\(^{27}\)

Comparing benchmark equilibrium economic variables corresponding to the three survival functions two things stand out. First, the aggregate equilibrium values for the Rectangular and BCL survival functions are generally quite close to each other, while the equilibrium corresponding to the Blanchard survival function is very far removed. This reflects its fundamental difference due to its convex rather than concave shape. It also suggests that the Rectangular survival function may in fact serve as a workable approximation to describing aggregate behavior.\(^{28}\) The second observation is that as we move from the Rectangular through the BCL to the Blanchard survival functions we are increasing the degree of uncertainty associated with survival. As a result agents save more, particularly as we move to the Blanchard case, where potentially an agent can live indefinitely. With a constant world interest rate but diminishing marginal productivity of labor, long-lived agents are able to accumulate their resources by investing abroad, enabling them to reduce their labor supply and domestic output, while increasing consumption.

7. Structural Changes

From the initial baseline equilibrium we analyze the transitional and long-run effects of the following structural changes: (i) a 25% increase in productivity \( A \), (ii) an increase in the tax rate on ordinary income from 10% to 15%, (iii) an increase in the world interest rate from 5% to 5.5%, and (iv) an increase in the rate of time preference from 3.5% to 4%.\(^{29}\) The long-run responses corresponding to the three demographic structures are reported in Table 2, while the dynamic adjustments of key variables, relative to their respective initial steady-state equilibria, are illustrated in Figs. 2.

\(^{27}\) The corresponding equilibrium values for the Rectangular distribution are \( \mu_c = 0.0129, \mu_{12} = 0.0027, \mu_3 = 0.0035 \). In the case of the Blanchard distribution they all remain constant throughout the transition, equal to 0.0128.

\(^{28}\) See also Bruce and Turnovsky (2013a) who conduct similar comparisons in the context of an endogenous growth model, although they consider the de Moivre function rather than the BCL function as the benchmark survival function. Azomahou, Boucekkine, and Diene (2009) use a standard endogenous growth model to compare the implications of the BCL and Blanchard survival functions for the relationship between life expectancy and economic growth.

\(^{29}\) An increase in the tax rate on interest income is comparable to that of a decrease in \( r^* \) and can be easily inferred.
7.1 Increase in productivity

In the long run, a 25% increase in productivity leads to a 25% increase in the holding of traded bonds, output, consumption, and human wealth. Leisure and the trade deficit expressed as a proportion of output, namely \((\bar{c} - \bar{y})/\bar{y}\), remain unchanged. These long-run adjustments are identical for all three survival functions. While these responses also dominate in the short run, there is a weak offsetting factor due to the higher productivity creating an immediate incentive to increase labor supply, thereby reducing leisure and reducing the increase in consumption, while augmenting the increase in output; see Figs. 2.A(d), 2.A(e), 2.A(c). Assuming that the economy starts out with a balance of payments equilibrium, this creates an immediate trade surplus, causing the economy to begin accumulating foreign bonds. Figs. 2.A(a), 2.(b). As the economy accumulates wealth its marginal utility declines, causing consumption and leisure to increase, slowing down the accumulation of traded bonds, and causing the economy to converge to its long-run equilibrium.

7.2 Increase in tax on income

Qualitatively, this structural change illustrated in Fig. 2.B, is approximately a mirror image of the productivity increase. In all cases, the long-run holdings of traded bonds, output, consumption, and human wealth all decline in the same proportions (approximately 2.8%), while leisure increases slightly (around 1%), and the relative trade deficit remains unchanged. But in contrast to the productivity increase the proportionality factors vary slightly between the three mortality rates. The long-run contraction is triggered by the fact that the higher tax rate reduces the after-tax real wage, causing an initial decrease in labor supply (increase in leisure) and income. With consumption and leisure being Edgeworth complements in utility, consumption declines by less than output, causing an immediate increase in the trade deficit so that the holdings of traded bonds begins to decline. This causes consumption to continue to decline, accompanied by declining leisure, leading to a partial reversal in output as the economy converges to its new steady state.
7.3 Increase in foreign interest rate

In all cases an increase in the foreign interest rate raises the long-run stock of traded bonds, increases leisure and consumption, reduces human wealth and output. In the short run an increase in the foreign interest rate by reducing the present discounted value of future wage income impacts the economy by reducing human wealth at birth. This reduces consumption and leisure and hence output increases, albeit slightly in all cases. The increase in interest income creates a current account surplus and the economy begins to accumulate traded bonds. This leads to an increase in consumption, accompanied by an increase in leisure leading to a decline in output. This causes a trade deficit, which however is dominated by the interest earned on the foreign bond holdings, causing an overall current account surplus and accumulation of traded bonds over time. In contrast to the first two shocks, there are some sharp differences in the long-run responses for the Blanchard survival function and the other two. For example, under Blanchard output declines by 5.67% almost three times the 2.0% reduction under BCL.

7.4 Increase in rate of time preference

This is essentially a mirror image of the foreign interest rate shock. In all cases an increase in the rate of time preference reduces the long-run stock of traded bonds, decreases leisure, consumption, and human wealth, but raises output. In the short run an increase in the rate of time preference impinges on the economy by raising the marginal propensity to consume. With human wealth remaining unaffected, this raises consumption and leisure thereby reducing output. The resulting trade deficit creates a current account deficit and the economy begins to decumulate traded bonds. This leads to a decline in consumption, accompanied by a decline in leisure leading to an increase in output. This leads to a trade surplus, which however is dominated by the decline in interest earned on the foreign bond holdings, causing an overall current account deficit and decumulation of traded bonds over time. Again, the long-run responses for the Blanchard survival function may deviate sharply. In this case, output increases by 5.87% as compared to 1.52% for BCL.

We may summarize the comparative responses to the four structural changes, under the alternative survival functions as follows. In all cases, the BCL and the Rectangular survival functions
track each other remarkably closely, both in absolute terms as well as in terms relative to their respective equilibria. In contrast, the Blanchard survival function deviates sharply in terms of its levels from both. For example, its implied level of output is 5.8% below that of the BCL, while its level of foreign bonds is 177% higher. In terms of the relative dynamics, the comparability between the Blanchard and other two depends upon the source of the structural change and how closely they interact with the demographic structure. In the case of the productivity increase and the general income tax increase, which interact only indirectly with the survival function, the dynamics relative to their respective equilibria of all three demographic structures follow one another closely. In contrast, for the interest rate and rate of time preference shocks the dynamics of the Blanchard survival function deviates substantially. This is because in either case the shock interacts directly with the mortality function via the time discounting element. With the Blanchard function substantially understating the survival rate for much of the distribution across cohorts, this magnifies the size of the corresponding shock relative to the two other survival functions.

8. Demographic Structure and Natural Rates of Wealth and Income Inequality

An early article by Atkinson (1971) suggests that the changing savings behavior of agents over their life-cycle generates an inherent wealth inequality. While this cannot be accommodated in the typical deterministic representative agent model, the overlapping generations structure of the demographic model, enables us to trace out the development of assets over the individual life-cycle. This is seen in Figure 3, where the asset path is hump-shaped over the life-cycle.\(^{30}\) Individuals begin with zero assets, then build up assets for intertemporal consumption smoothing and, toward the end of their lives, deplete their assets so as to assure that assets are zero exactly at the maximum attainable life time, \(D\).

The fact that individuals at different stages of their life-cycle possess different levels of wealth, knowing the size of the various cohorts enables us to calculate standard wealth inequality measures, such as the Gini coefficient. This measure indicates the degree of inequality inherent in an economy purely due its age composition and abstracts from any within-cohort inequality arising from

\(^{30}\) We note that with idiosyncratic stochastic income shocks wealth inequality can also be generated by the representative agent model.
differential endowments or skill levels. In this sense, it can be termed the “natural rate of wealth inequality”.

The first row of Table 3 reports the Gini coefficient of wealth inequality for the benchmark parameterization corresponding to the three survival functions. They increase from 0.278 for the BCL through 0.289 for the Rectangular to 0.428 for the Blanchard survival function. While the BCL and Rectangular functions are very close, the substantially larger proportion of old agents in the Blanchard economy means more very old wealthy people and therefore more wealth inequality. That these estimates are well below the US wealth Gini coefficient of 0.80 is to be expected, since it is reflecting only one factor influencing wealth inequality, namely the demographic composition.

In addition to generating aggregate dynamics, the demographic structure has consequences for the distributional dynamics. Rows 2-6 of the upper panel report the long-run changes in the wealth Gini coefficients in response to various structural changes. Thus while an increase in productivity raises wealth of all age groups, it does so in a neutral fashion, so that the Gini remains unchanged. Likewise raising the general income tax rate, \( r_y \), from 10% to 15% reduces proportionately the wealth of all groups again leaving the Gini unchanged. In contrast, raising the return to traded bonds by benefiting more the older agents with more wealth raises wealth inequality, while increasing the rate of time preference reduces the wealth Gini.

The lower panel of Table 3 reports the corresponding income Gini coefficients. The point here is that the impact of the demographic structure on income inequality is almost negligible. This reflects two elements of the model. First, all agents regardless of age receive the same wage rate; second profit income is distributed uniformly across cohorts. The only element of income that varies across cohorts is labor supply. Since this variation is also limited, the overall variation in income across age groups is very small, as Fig. 3 clearly illustrates.

---

31 Mierau and Turnovsky (2014a) calculate a similar measure for an endogenous growth model of a closed economy.

32 Benhabib, Bisin, and Zhu (2014) examine the dynamics of the distribution of wealth for the Blanchard survival function and show that with idiosyncratic investment shocks the wealth follows a double Pareto distribution. Our objective is somewhat different, namely to show how a plausible demographic structure can break the knife edge in an otherwise deterministic model. It is well known that the presence of stochastic rates of return offers a further alternative to breaking the knife edge; see e.g. Grinols and Turnovsky (1994). Thus an interesting extension of the present framework would be to characterize the nature of the wealth distribution when stochastic elements are added to the more realistic BCL demographic structure.
9. Conclusions

Closing the small open economy model has been a stumbling block in studying the dynamic evolution of such models. The typical procedure of equating the after-tax return on traded bonds to the rate of time preference involves imposing a knife-edge condition, which severely constrains the ability to address the dynamic characteristics of small open economy models. Existing procedures to circumvent these difficulties involve endogenizing either the rate of time preference or the rate of return on traded bonds. While both approaches may resolve the problem, each is associated with its own set of issues. Endogenizing the rate of time preference typically leads to implausible equilibrium dynamics, while endogenizing the rate of return is essentially dispensing with an important aspect of the small open economy.

In this paper we have adopted an alternative approach, namely replacing the infinitely-lived representative agent framework with a plausible demographic structure. Introducing a general survival function enables us to derive a well-behaved plausible macrodynamic equilibrium without imposing any restrictive knife-edge relationship between rates of return and preferences. We have characterized the equilibrium and traced out the dynamics in response to several alternative classes of structural change. Two general conclusions stand out. First, for our plausible parameterization the simple Rectangular survival function, characteristic of the seminal Samuelson-Diamond model, tracks more general survival functions, estimated from real demographic data, surprisingly closely, both in levels and in its relative dynamics. Second, the Blanchard survival function tracks the data poorly in terms of absolute levels, while the closeness of its relative dynamics is dependent on the source of the structural change.

Finally, by focusing on a realistic demographic structure as an appealing way to close the small open economy model, this model has not been developed to match observed data trends for anything other than the age-dependent survival probabilities, and therefore is highly stylized in all other dimensions. In particular, we have simplified the production structure insofar as possible. While this is an obvious limitation, we nevertheless believe that the model suffices to make our point in a transparent way. As noted, the small open economy model may be closed using simpler ad-hoc
techniques such as cost of adjustment functions and unrealistic overlapping-generations structures. However these approaches are arbitrary and lack any firm theoretical underpinning. With this paper we attempt to develop an approach that both simultaneously relaxes the knife-edge constraint, and increases the realism of the basic framework. While it is true that it may increase the complexity of the modeling environment, we find the benefits from removing the knife-edge restrictions in this realistic way to be quite compelling.

At the same time, clearly an important extension of this framework is to introduce physical capital. Expanding the model in this direction will enable us to subject the significance of the demographic structure to empirical testing, something that the present model is too stylized to do effectively. It will also enable us to address a current issue of importance, specifically the impact of differential demographic structures on the international flow of capital; see Ghironi (2006), Backus et al. (2014). The approach developed in the present paper promises to provide a fruitful one for further investigation of this topical issue.
### Table 1: Baseline Parameter and Equilibrium Values

<table>
<thead>
<tr>
<th>Structural Parameters</th>
<th>Value:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total factor productivity, $A$</td>
<td>1</td>
</tr>
<tr>
<td>Labor share of output, $\alpha$</td>
<td>0.65</td>
</tr>
<tr>
<td>Intertemporal Elasticity of Substitution, $\sigma = 1/(1 - \gamma)$</td>
<td>0.40</td>
</tr>
<tr>
<td>Elasticity of leisure in utility, $\theta$</td>
<td>1.75</td>
</tr>
<tr>
<td>Pure rate of time preference, $\rho$</td>
<td>3.5%</td>
</tr>
<tr>
<td>World interest rate, $r$</td>
<td>5.0%</td>
</tr>
<tr>
<td>Population growth rate, $n$</td>
<td>1.0%</td>
</tr>
<tr>
<td>Distortionary Tax on Wage Income, $\tau_y$</td>
<td>10.0%</td>
</tr>
<tr>
<td>Distortionary Tax on Interest Income, $\tau_r$</td>
<td>10.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Demographic Parameters:</th>
<th>Rectangular</th>
<th>BCL</th>
<th>Blanchard</th>
</tr>
</thead>
<tbody>
<tr>
<td>Life expectancy at age 18, $L_{18}$</td>
<td>78.38</td>
<td>78.38</td>
<td>78.38</td>
</tr>
<tr>
<td>Implied maximum age, $D$</td>
<td>78.38</td>
<td>95.06</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Time pref. of average individual $\rho + \mu(\bar{u})$</td>
<td>3.50%</td>
<td>3.88%</td>
<td>4.78%</td>
</tr>
<tr>
<td>Youth mortality, $\mu_0$</td>
<td>0</td>
<td>78.3618 (6.0193)</td>
<td>N/A</td>
</tr>
<tr>
<td>Old age mortality, $\mu_1$</td>
<td>0</td>
<td>0.0566 (0.0011)</td>
<td>0.01275</td>
</tr>
<tr>
<td>Birth Rate (Implied), $\beta$</td>
<td>2.21%</td>
<td>2.24%</td>
<td>2.28%</td>
</tr>
<tr>
<td>Lifetime consumption growth rate</td>
<td>0.20%</td>
<td>0.20%</td>
<td>0.20%</td>
</tr>
</tbody>
</table>

Numbers in parentheses are standard errors

<table>
<thead>
<tr>
<th>Implied Economic Variables:</th>
<th>Rectangular</th>
<th>BCL</th>
<th>Blanchard</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output, $\hat{y}$</td>
<td>0.395</td>
<td>0.394</td>
<td>0.371</td>
</tr>
<tr>
<td>Traded Bonds, $\tilde{f}$</td>
<td>0.592</td>
<td>0.682</td>
<td>1.89</td>
</tr>
<tr>
<td>Relative trade deficit, $(\hat{c} - \hat{y}) / \hat{y}$</td>
<td>0.0599</td>
<td>0.0693</td>
<td>0.203</td>
</tr>
<tr>
<td>Leisure, $\bar{l}$</td>
<td>0.760</td>
<td>0.762</td>
<td>0.783</td>
</tr>
<tr>
<td>Wage rate, $\hat{w}$</td>
<td>1.071</td>
<td>1.074</td>
<td>1.110</td>
</tr>
<tr>
<td>Human Wealth, $\bar{H}$</td>
<td>8.54</td>
<td>8.32</td>
<td>7.31</td>
</tr>
<tr>
<td>Consumption, $\hat{c}$</td>
<td>0.419</td>
<td>0.421</td>
<td>0.446</td>
</tr>
<tr>
<td>Marginal Propensity to Consume, $1/\Lambda$</td>
<td>0.0465</td>
<td>0.0478</td>
<td>0.0558</td>
</tr>
<tr>
<td>Stable Eigenvalue, $\Lambda$</td>
<td>-0.0188</td>
<td>-0.0191</td>
<td>-0.0184</td>
</tr>
</tbody>
</table>
Table 2: Long-run Effects of Permanent Structural Changes

A. Rectangular survival function

<table>
<thead>
<tr>
<th></th>
<th>Traded bonds</th>
<th>Output</th>
<th>Leisure</th>
<th>Consumption</th>
<th>Human wealth</th>
<th>Relative Trade Deficit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Equilibrium</td>
<td>0.592</td>
<td>0.395</td>
<td>0.760</td>
<td>0.419</td>
<td>8.54</td>
<td>0.0599</td>
</tr>
<tr>
<td>Increase in A from 1</td>
<td>0.740</td>
<td>0.494</td>
<td>0.760</td>
<td>0.524</td>
<td>10.67</td>
<td>0.0600</td>
</tr>
<tr>
<td>Increase in $y$</td>
<td>0.575</td>
<td>0.384</td>
<td>0.771</td>
<td>0.407</td>
<td>8.30</td>
<td>0.0599</td>
</tr>
<tr>
<td>Increase in $r$</td>
<td>0.853</td>
<td>0.388</td>
<td>0.767</td>
<td>0.427</td>
<td>7.95</td>
<td>0.0990</td>
</tr>
<tr>
<td>Increase in $\rho$</td>
<td>0.292</td>
<td>0.401</td>
<td>0.755</td>
<td>0.410</td>
<td>8.49</td>
<td>0.3473</td>
</tr>
</tbody>
</table>

B. BCL Survival function

<table>
<thead>
<tr>
<th></th>
<th>Traded bonds</th>
<th>Output</th>
<th>Leisure</th>
<th>Consumption</th>
<th>Human wealth</th>
<th>Relative Trade Deficit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Equilibrium</td>
<td>0.682</td>
<td>0.394</td>
<td>0.762</td>
<td>0.421</td>
<td>8.32</td>
<td>0.0693</td>
</tr>
<tr>
<td>Increase in A from 1</td>
<td>0.852</td>
<td>0.492</td>
<td>0.762</td>
<td>0.526</td>
<td>10.40</td>
<td>0.0693</td>
</tr>
<tr>
<td>Increase in $y$</td>
<td>0.662</td>
<td>0.383</td>
<td>0.772</td>
<td>0.409</td>
<td>8.09</td>
<td>0.0690</td>
</tr>
<tr>
<td>Increase in $r$</td>
<td>0.979</td>
<td>0.386</td>
<td>0.769</td>
<td>0.430</td>
<td>7.76</td>
<td>0.1142</td>
</tr>
<tr>
<td>Increase in $\rho$</td>
<td>0.335</td>
<td>0.400</td>
<td>0.756</td>
<td>0.414</td>
<td>8.27</td>
<td>0.0334</td>
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</tbody>
</table>

C. Blanchard Survival Function

<table>
<thead>
<tr>
<th></th>
<th>Traded bonds</th>
<th>Output</th>
<th>Leisure</th>
<th>Consumption</th>
<th>Human wealth</th>
<th>Relative Trade Deficit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Equilibrium</td>
<td>1.89</td>
<td>0.371</td>
<td>0.783</td>
<td>0.446</td>
<td>7.31</td>
<td>0.203</td>
</tr>
<tr>
<td>Increase in A from 1</td>
<td>2.36</td>
<td>0.464</td>
<td>0.783</td>
<td>0.560</td>
<td>9.14</td>
<td>0.203</td>
</tr>
<tr>
<td>Increase in $y$</td>
<td>1.83</td>
<td>0.360</td>
<td>0.792</td>
<td>0.433</td>
<td>7.10</td>
<td>0.203</td>
</tr>
<tr>
<td>Increase in $r$</td>
<td>2.72</td>
<td>0.350</td>
<td>0.801</td>
<td>0.470</td>
<td>6.95</td>
<td>0.350</td>
</tr>
<tr>
<td>Increase in $\rho$</td>
<td>0.880</td>
<td>0.390</td>
<td>0.765</td>
<td>0.420</td>
<td>7.16</td>
<td>0.091</td>
</tr>
</tbody>
</table>
Table 3:
GINI Coefficients for Wealth and Income

<table>
<thead>
<tr>
<th></th>
<th>Wealth GINI Coefficients:</th>
<th>Income GINI Coefficients:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BCL</td>
<td>Rectangular</td>
</tr>
<tr>
<td>Initial Equilibrium</td>
<td>0.278</td>
<td>0.289</td>
</tr>
<tr>
<td>A=1 to A=1.25</td>
<td>0.278</td>
<td>0.289</td>
</tr>
<tr>
<td>r=5% to 5.5%</td>
<td>0.284</td>
<td>0.293</td>
</tr>
<tr>
<td>ρ=3.5% to ρ=4%</td>
<td>0.277</td>
<td>0.288</td>
</tr>
<tr>
<td>τ_r=10% to τ_r=15%</td>
<td>0.275</td>
<td>0.287</td>
</tr>
<tr>
<td>τ_y=10% to τ_y=15%</td>
<td>0.278</td>
<td>0.289</td>
</tr>
</tbody>
</table>
Figure 1:
Survival functions per demography

Based on 2006 US data from www.mortality.org
Figure 2.A
Per-capita Transition Paths – Increase in Productivity from \( A = 1 \) to \( A = 1.25 \)

Figure 2.B
Per-capita Transition Paths – Increase in Income tax from \( \tau_y = 10\% \) to \( \tau_y = 15\% \)
Figure 2.C
Per-capita Transition Paths – Increase in Foreign Interest Rate from r=5% to r=5.5%

Figure 2.D
Per-capita Transition Paths – Increase in Rate of Time Preference from $\rho=3.5\%$ to $\rho=4\%$
Figure 3
Distribution of assets and income over ages by shock

Where: A represents the productivity shock, r the interest rate shock, ρ the time pref. shock, τy the income tax shock.
Appendix

A. Derivation of Macrodynmic Equilibrium in the Demographic Economy

Setting \( v = t \) in (10a), and recalling \( F(t, t) = 0 \), we see that the consumption of newborns, \( C(t, t) \), involves the dynamics of their human wealth and marginal propensity to consume

\[
C(t, t) = \frac{H(t)}{\Delta(t)} \tag{A.1a}
\]

where for notational convenience (and using (6a)) we write

\[
H(t) \equiv H(t, t) = \int_t^{t+D} \left[ (1 - \tau_y) [w(\tau) + \Pi(\tau)] - \theta C(t, \tau) - T(\tau) \right] e^{-(1 - \tau_y) \mu(\tau)} d\tau \tag{A.1b}
\]

\[
\Delta(t) \equiv \Delta(t, t) = \int_t^{t+D} e^\int_t^\tau \left[ H(s) \right] ds \left[ H(s) \right] ds \tag{A.1c}
\]

Differentiating (A.1b) with respect to \( t \) yields

\[
\dot{H}(t) = -(1 - \tau_y) [w(t) + \Pi(t)] + \theta C(t, t) + T(t) + (1 - \tau_y) \mu H(t) \tag{A.2}
\]

\[
+ \int_t^{t+D} \mu(\tau - t) [(1 - \tau_y) [w(\tau) + \Pi(\tau)] - \theta C(t, \tau) - T(\tau)] e^{-(1 - \tau_y) \mu(\tau - t)} d\tau

- \theta \int_t^{t+D} C(t, \tau) e^{-(1 - \tau_y) \mu(\tau - t)} d\tau
\]

Combining (A.1a) with (8) yields

\[
C(t, \tau) = C(t, t) e^\int_t^\tau H(s) ds = \left( \frac{H(t)}{\Delta(t)} \right) e^\int_t^\tau H(s) ds
\]

and hence

\[
C(t, \tau) = \left[ \frac{d}{dt} \left( \frac{H(t)}{\Delta(t)} \right) - \left( \frac{H(t)}{\Delta(t)} \right) \psi(t) \right] e^\int_t^\tau H(s) ds
\]

Utilizing the definition of \( \Delta(t) \), we may write

\[
\int_t^{t+D} C(t, \tau) e^{-(1 - \tau_y) \mu(\tau - t)} d\tau = \left[ \frac{d}{dt} \left( \frac{H(t)}{\Delta(t)} \right) - \left( \frac{H(t)}{\Delta(t)} \right) \psi(t) \right] \int_t^{t+D} e^\int_t^\tau H(s) ds H(s) ds
\]

29
\[
\frac{d}{dt} \left( \frac{H(t)}{\Delta(t)} \right) - \frac{H(t)}{\Delta(t)} \psi(t) \Delta(t)
\]

and thus

\[
\int_{t}^{t+D} C(t, \tau) e^{-(1-\tau_y)\nu_y(\tau-t)-M(\tau-t)} d\tau = \frac{\dot{H}(t)}{\Delta(t)} \hat{\Delta}(t) - \psi(t) H(t) \tag{A.3}
\]

Also, we define

\[
\mu_H (\tau_1 - t) = \frac{\int_{t}^{t+D} \mu(\tau-t)[(1-\tau_y)\{w(\tau) + \Pi(\tau)\}-\theta C(t, \tau)-T(\tau)]e^{-(1-\tau_y)\nu_y(\tau-t)-M(\tau-t)} d\tau}{\int_{t}^{t+D} [(1-\tau_y)\{w(\tau) + \Pi(\tau)\}-\theta C(t, \tau)-T(\tau)]e^{-(1-\tau_y)\nu_y(\tau-t)-M(\tau-t)} d\tau}
\]

\[
= \frac{1}{H(t)} \int_{t}^{t+D} \mu(\tau-t)[(1-\tau_y)\{w(\tau) + \Pi(\tau)\}-\theta C(t, \tau)-T(\tau)]e^{-(1-\tau_y)\nu_y(\tau-t)-M(\tau-t)} d\tau \quad \tau_1 \in (t, t+D) \tag{A.4}
\]

Thus \( \mu_H (\tau_1 - t) \) is the ratio of the human wealth given up by the dying to aggregate human wealth and can be interpreted as providing an estimate of average mortality over the period \((t-D, t)\), from information on human wealth across the cohorts. Then substituting (A.3) and (A.4) into (A.2) yields

\[
(1+\theta)\dot{H}(t) - \frac{\theta H(t)}{\Delta(t)} \Delta(t) = -\{(1-\tau_y)\{w(\tau) + \Pi(\tau)\}+\theta H(t)\Delta(t) + T(t) + \{(1-\tau_y)\nu_y(\tau-t)-M(\tau-t)\}H(t) \tag{A.5}
\]

Finally, differentiating (A.1c) with respect to \( t \) yields the dynamics of the marginal propensity to consume, namely

\[
\hat{\Delta}(t) = -1 + \{(1-\tau_y)\nu_y + \mu_a - \psi(t)\} \Delta(t) \tag{A.6}
\]

where analogously we define:

\[
\mu_a (\tau_2 - t) = \frac{1}{\Delta(t)} \int_{t}^{t+D} \mu(\tau-t)e^{\nu_z(\tau-t)-M(\tau-t)} d\tau \quad \tau_2 \in (t, t+D) \tag{A.7}
\]

Thus equations (16), (23), (A.6), and (A.7), which specify the evolution of per capita consumption, \( c(t) \), per capita holdings of foreign assets, \( f(t) \), together with human capital at birth, \( H(t) \), and the corresponding marginal propensity to consume, \( \Delta(t) \), describe the core equilibrium macroeconomic dynamics in the demographic small open economy. However, these relationships
involve: (i) the equilibrium wage $w(t)$ and its growth rate, $\dot{w}(t)/w(t)$, (ii) equilibrium leisure, $l(t)$, (iii) equilibrium lump-sum taxes, $T(t)$, and (iv) the individuals’ growth rate of consumption, $\psi(t)$. Thus the complete aggregate equilibrium needs to take account of (7), (19), (20), (21), (22a), and (22b). Omitting details, the macrodynamic equilibrium can be expressed by the following autonomous fourth order system

$$
\begin{pmatrix}
1 + \frac{\theta \gamma}{1 - \gamma(1 + \theta)} \left[ l(c)(1 - \alpha) \right] & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{\theta^2 \gamma}{1 - \gamma(1 + \theta)} \left[ \frac{l(c)(1 - \alpha)}{1 - \alpha l(c)} \right] \frac{H}{c} & 0 & 1 + \theta - \frac{\theta H}{\Delta} & \dot{f}(t) \\
\frac{\theta \gamma}{1 - \gamma(1 + \theta)} \left[ \frac{l(c)(1 - \alpha)}{1 - \alpha l(c)} \right] \frac{\Delta}{c} & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{c}(t) \\
\dot{f}(t) \\
\dot{H}(t) \\
\dot{\Delta}(t)
\end{pmatrix}
$$

(A.8)

where $l(c)$ is obtained by solving the equation obtained by combining (20) and (21)

$$
l = \frac{\theta c}{(1 - \tau_r)A\alpha(1 - l)^{a - 1}}
$$

so that

$$
l_c = \frac{\partial l}{\partial c} = \frac{1 - l(c)}{c \left[ 1 - \alpha l(c) \right]} > 0
$$

Linearizing (25) around the steady-state values $(\tilde{c}, \tilde{f}, \tilde{H}, \tilde{\Delta})$, the local dynamics are

$$
\begin{pmatrix}
1 + \frac{\theta \gamma}{1 - \gamma(1 + \theta)} \left[ l(\tilde{c})(1 - \alpha) \right] & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{\theta^2 \gamma}{1 - \gamma(1 + \theta)} \left[ \frac{l(\tilde{c})(1 - \alpha)}{1 - \alpha l(\tilde{c})} \right] \frac{\tilde{H}}{\tilde{c}} & 0 & 1 + \theta - \frac{\theta \tilde{H}}{\Delta} & \dot{\tilde{f}}(t) \\
\frac{\theta \gamma}{1 - \gamma(1 + \theta)} \left[ \frac{l(\tilde{c})(1 - \alpha)}{1 - \alpha l(\tilde{c})} \right] \frac{\Delta}{\tilde{c}} & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{\tilde{c}}(t) \\
\dot{\tilde{f}}(t) \\
\dot{\tilde{H}}(t) \\
\dot{\tilde{\Delta}}(t)
\end{pmatrix}
$$

(A.9)
This is the system employed to analyze the local dynamics.

B. Mortality-specific functions

Here we substitute the three mortality function into (27b)-(27d) to yield the specific expressions for $\tilde{H}, \tilde{\Delta}, \tilde{c}$:

**Human wealth at birth:**

$$
\tilde{H} = \frac{\tilde{\gamma} + \tilde{w}\tilde{l}(1-\tau_y) + \tau_y r^* \tilde{f}}{(1-\mu) \mu_{0-1} - \mu_0} - \frac{1}{e^{-\mu_0}} \left(1-e^{-(1-\tau_y)\mu_0}\right)
$$

BCL:

$$
-\theta \frac{\tilde{H}}{\Delta} \frac{1}{\mu_0 - 1} \left(\begin{array}{c}
\frac{\mu_0}{1-\tau_y} \left(1-e^{-\left(1-\tau_y\right)\mu_0}\right) - \frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right) \\
\mu_0 - 1 \left(1-e^{-(1-\tau_y)\mu_0}\right) - \frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right)
\end{array}\right)
$$

Rectangular: $\tilde{H} = \frac{\tilde{\gamma} + \tilde{w}\tilde{l}(1-\tau_y) + \tau_y r^* \tilde{f}}{(1-\tau_y)r^* - \tilde{\psi}} - \theta \frac{\tilde{H}}{\Delta} \left(\begin{array}{c}
\frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right) \\
\frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right)
\end{array}\right)$

Blanchard: $\tilde{H} = \frac{\tilde{\gamma} + \tilde{w}\tilde{l}(1-\tau_y) + \tau_y r^* \tilde{f}}{(1-\tau_y)r^* + \mu_b - \tilde{\psi}} - \theta \frac{\tilde{H}}{\Delta} \left(\begin{array}{c}
\frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right) \\
\frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right)
\end{array}\right)$

**Inverse of the MPC:**

BCL: $\tilde{\Delta} = \frac{1}{\mu_0 - 1} \left(\begin{array}{c}
\frac{\mu_0}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right) - \frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right) \\
\mu_0 - 1 \left(1-e^{-(1-\tau_y)\mu_0}\right) - \frac{1}{1-\tau_y} \left(1-e^{-(1-\tau_y)\mu_0}\right)
\end{array}\right)$

Rectangular: $\tilde{\Delta} = \frac{1}{(1-\tau_y)r^* - \tilde{\psi}} \left(1-e^{-(1-\tau_y)\mu_0}\right)$

Blanchard: $\tilde{\Delta} = \frac{1}{(1-\tau_y)r^* + \mu_b - \tilde{\psi}}$
Per-capita consumption:

BCL: \[
\bar{c} = \beta \frac{\bar{H}}{\Delta} \left( \frac{1}{\mu_0 - 1} \right) \left( \frac{1}{n - \psi} \left[ 1 - e^{(\psi - \eta)\bar{D}} \right] - \frac{1}{n - \psi} - \mu_i \left[ 1 - e^{(\psi - n + \mu_i)\bar{D}} \right] \right)
\] (A.12a)

Rectangular: \[
\bar{c} = \beta \frac{\bar{H}}{\Delta} \left( \frac{1}{n - \psi} \right) \left( 1 - e^{(\psi - \eta)\bar{D}} \right)
\] (A.12b)

Blanchard: \[
\bar{c} = \beta \frac{\bar{H}}{\Delta} \left( \frac{1}{n + \mu_i - \psi} \right)
\] (A.12c)
References


overlapping generations,” *Macroeconomic Dynamics* 17, 1605-1637.


