Suitable-Portfolio Investors, Nondominated Frontier Sensitivity, and the Effect of Multiple Objectives on Standard Portfolio Selection

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Abstract

In standard portfolio theory, an investor's only goal is to pursue a portfolio return maximization strategy. But in this paper, we consider another type of investor whose goal is to build, more broadly, a *suitable portfolio* taking additional factors into account. For instance, in addition to portfolio return, such an investor may wish to monitor his or her portfolio with regard to the proportion of portfolio return derived from dividends, the maximum amount invested in any security, short selling, the number of securities involved, social responsibility, and so forth. To accommodate such an investor, we develop a multiple criteria portfolio selection formulation, corroborate its appropriateness by examining the sensitivity of the nondominated frontier to various factors, and observe the conversion of the nondominated *frontier* to a nondominated *surface*. Furthermore, multiple criteria enable us to provide an explanation as to why the "market portfolio," so often found deep below the nondominated frontier, is roughly where one would expect it to be with multiple criteria. After commenting on approaches for searching nondominated surfaces, the paper concludes with the idea that what is the "modern portfolio theory"

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of today might best be interpreted as a projection onto two-space of the real multiple criteria portfolio selection problem from higher dimensional space.

1 Introduction

On one hand, we have the portfolio selection problem in finance, its Markowitz [33, 35, 36] mean-variance formulation, and the extent to which this formulation has permeated the field of finance for the past half century. On the other, there is the trickle of multiple criteria oriented portfolio papers that has increased from about 1.5 per year over the period 1973-1992 to about 4.5 per year since 1993 (as analyzed from Steuer and Na [45]).

The question is this: Is the growing trickle of no especial significance, or might the multiple criteria oriented papers be indicative of a new brand of portfolio thinking appearing on the horizon? We feel that there are valuable thought processes stirring among these multiple criteria papers. Of particular interest is the proposition that all investors are not the same and that different categories of investors have different modeling needs, currently not well met by approaches from the standard portfolio theory community. It is toward the idea that all investors cannot be put into the same category, its implications, and what can be done for different investors, that this paper is directed.

Clearly, the amount of research that has been devoted to portfolio theory is remarkable. But knowing that all kinds of investors exist in the real world, one difficulty with existing theory is that it only assumes a *standard* investor. By "standard" investor we are referring to investors whose behavior complies with the conventional assumptions of finance and who are only in it for the money, i.e., to maximize portfolio return. Notice that we just said "portfolio return", not "expected portfolio return." We will elaborate on this shortly.

But in this paper, not to say that there aren't large numbers of investors for which the definition of standard is apt, we wish to consider, for side-by-side existence with standard investors, another type of investor. To this other type of investor we ascribe the designation *suitable-portfolio* investor. This is to signify that the goal of this type of investor, beyond merely the monetary, is to strive for the construction of a "suitable portfolio." For instance, consider a parent who wishes to create a portfolio for his or her child to interest the child in investing. Would the parent purchase stock in tobacco, alcohol, or cluster bomb manufacturing companies? Probably not. Most likely the parent would seek stocks like Disney, McDonald's and Merck. Additionally, the parent might wish to limit the number of securities to keep the child from losing focus and overweight in stocks that pay dividends to generate a warm reminder for the child of the project every three months. Although only a semi-realistic example, the intention here is to indicate that suitable-portfolio investing is not an irrational proposition and that there may well be a little, if not a lot, of suitable-portfolio investing in all of us.

Consequently, this means that many investors may well wish to seek balance in their portfolios among various competing concerns such as dividends, the maximum amount invested in any security, short selling, the number of securities in a portfolio, social responsibility, and so forth. Noting that (a) existing portfolio theory is still firmly grounded in approaches developed over 50 years ago, (b) the world has almost certainly become more complex in the interim, and (c) new tools such as those from multiple criteria decision analysis have come online in the meantime, with evidence mounting, it now appears timely to see if existing portfolio theory can't be extended to better meet the modeling needs of investors other than of the standard variety.

To place previous multiple criteria oriented portfolio analysis research into perspective, six categories are employed: (1) overall framework, (2) portfolio ranking, (3) skewness inclusion, (4) use of alternative measures of risk, (5) decision support systems, and (6) the modeling of individual investor preferences.

In the first category, we have overview pieces such as by Hallerbach and Spronk [22, 23] and Bana e Costa and Soares [5] in which the benefits of embracing multiple criteria concepts in financial decision making are outlined. Employing tools (see Belton and Stewart [7]) from multiple criteria decision analysis for portfolio ranking, we have papers represented by Hurson and Zopounidis [25], Yu [48], Jog, Kaliszewski and Michalowski [26], Hurson and Ricci [24], and Bouri, Martel and Chabchoub [9]. Noteworthy in the category of skewness inclusion are the papers by Stone [46], Konno, Shirakawa and Yamazaki [27], and Konno and Suzuki [28].

With regard to alternative measures of risk, we have the efforts by Zeleny [49], Konno and Yamazaki [29], Feinstein and Thapa [18], Doumpos, Spanos and Zopounidis [15], and Michalowski and Ogryczak [39]. With regard to decision support systems employing mathematical programming techniques, we have the approaches of Colson and DeBruyn [12], Ballestero and Romero [4], Tamiz, Hasham, Jones, Hesni and Fargher [47], Dominiak [14], Ogryczak [41], Arenas-Parra, Bilbao-Terol and Rodriguez-Uria [1], Ballestero and Pla-Santamaria [3], Mansini, Ogryczak and Speranza [32], Ehrgott, Klamroth and Schwehm [16], and Zopounidis and Doumpos [50]. In the sixth category, respecting the situation that some criteria may come from financial-economic theory and others may come from the individual investor, we have Spronk and Hallerbach [43], Ballestero [2], Chang, Meade, Beasley and Sharaiha [11], Gilli and Këllezi [20], and Bana e Costa and Soares [6]. It is mostly in this last category, in the area of recognizing the existence of investors other than standard investors, and in developing more inclusive approaches to portfolio selection, that this paper is concerned.

The paper proceeds as follows. In Section 2 we study the line of thinking that has led to the widespread acceptance of the mean-variance approach of standard portfolio selection, and in Section 3 we extend the line of thinking for suitable-portfolio investing. In Section 4 we review the topic of multiple criteria optimization, and in Section 5 we comment on the two most popular models of standard portfolio selection, the unrestricted-variable model which can be addressed in closed-form, and the variable-restricted model that requires mathematical programming. With standard portfolio theory under siege, the assumptions of "modern portfolio analysis" are examined in Section 6, and in Section 7 the sensitivity of the nondominated frontier to various factors is discussed. With a revised set of assumptions and the presentation of an expanded formulation in Section 8, the conversion of the nondominated frontier to a nondominated surface is outlined in Section 9. Computational viability is addressed in Section 10 and concluding remarks constitute Section 11.

2 Standard Theory

The problem of standard portfolio selection is as follows. Assume

(a) n securities
(b) an initial sum of money to be invested
(c) the beginning of a holding period
(d) the end of the holding period

and let $x_1, \ldots, x_n$ be the investor's *investment proportion* weights. These are the proportions of the initial sum to be invested in the n securities at the beginning of the holding period that define the portfolio to be held fixed until the end of the holding period. By standard theory, as in all of investing, there is only one motivating factor in portfolio selection. And that is to make money, i.e., to maximize portfolio return, the percent return earned by the portfolio over the course of the holding period. Reflecting this, the problem of standard portfolio selection is then to solve

$$\max\{\sum_{j=1}^{n} r_j x_j = R(\mathbf{x},\mathbf{r})\} \tag{1}$$
$$s.t. \quad \mathbf{x} \in S = \{\mathbf{x} \in R^n \mid \sum_{i=1}^{n} x_i = 1,\ \alpha_i \leq x_i \leq \omega_i\}$$

in which the r_j are the percent returns to be realized on the n securities over the holding period, the α_j and ω_j are fixed lower and upper bounds on the x_j, and S as specified in (1) with linear constraints is a typical feasible region. Unfortunately, to solve (1) is not so easy. The difficulty is that the x_j must be solved for at the beginning of the holding period, but the r_j are not known until the end of the holding period. This makes (1) a *stochastic* optimization problem, and what it means to solve a stochastic program requires an interpretation (for example, see Caballero, Cerdá, Muñoz, Rey and Stancu-Minasian [10]). To operationalize what it means to solve (1), there are several ways to proceed. The popular approach is the mean-variance approach which is justified as follows.

Let $U\colon D \to R$ represent the utility function of a standard investor where R is the real line. Since a standard investor's only purpose is to maximize portfolio return, U is then a function of only the single argument $R(\mathbf{x},\mathbf{r})$, and it suffices for the utility function's domain D to be given by the interval $D = [R_{min}, R_{max}]$

where

$$R_{min} = \min_{1 \le j \le n} \{r_{j\alpha}\} \qquad\qquad R_{max} = \max_{1 \le j \le n} \{r_{j\omega}\}$$

in which $r_{j\alpha}$ and $r_{j\omega}$ are lower and upper bounds on all possible realizations of r_j. Since standard investors are assumed to possess declining marginal utility, $U\colon D \to R$ must then be strictly concave and increasing. This is what is known as *risk aversion*. Often U is additionally assumed to be infinitely-differentiable so that, except in some pathological cases, U is Taylor-series expandable.

Since (1) is stochastic and what it means to solve a stochastic programming problem is not well-defined, let us consider what might be the interpretation with such a U if it is accepted that any $\mathbf{x} \in S$ that maximizes the expected utility of $R(\mathbf{x},\mathbf{r})$ solves (1), i.e., any $\mathbf{x}$ that solves the *substitute* problem

$$\begin{aligned} &\max\{E[U(R(\mathbf{x},\mathbf{r}))]\} \\ &s.t. \quad \mathbf{x} \in S \end{aligned} \tag{2}$$

solves (1). Now if in addition to exhibiting risk aversion U is quadratic, U would be of the form

$$U(R(\mathbf{x},\mathbf{r})) = a_0 + a_1 R(\mathbf{x},\mathbf{r}) + a_2 R(\mathbf{x},\mathbf{r})^2 \tag{3}$$

where $\alpha_2 < 0$ (as a result of the strict concavity of U). Taking the expectation of (3), we have

$$\begin{aligned} E[U(R(\mathbf{x},\mathbf{r}))] &= a_0 + a_1 E[R(\mathbf{x},\mathbf{r})] + a_2 E[R(\mathbf{x},\mathbf{r})^2] && (4)\\ &= a_0 + a_1 E[R(\mathbf{x},\mathbf{r})] + a_2 (V[R(\mathbf{x},\mathbf{r})] + E[R(\mathbf{x},\mathbf{r})]^2) && (5)\\ &= a_0 + a_1 E[R(\mathbf{x},\mathbf{r})] + a_2 E[R(\mathbf{x},\mathbf{r})]^2 + a_2 V[R(\mathbf{x},\mathbf{r})] && (6)\\ &= U(E[R(\mathbf{x},\mathbf{r})]) + a_2 V[R(\mathbf{x},\mathbf{r})] && (7) \end{aligned}$$

in which V designates variance. Equation (5) follows from (4) because the expected value of the square of a random variable is the variance plus the square of the expected value; and (7) follows from (6) by the re-substitution of $E[R(\mathbf{x},\mathbf{r})]$ in (3).

In (7), with U strictly concave and increasing, it is seen that all expected utility function contours are strictly convex and increasing in (variance, expected-return) space. Thus, with $E[U(R(\mathbf{x},\mathbf{r}))]$ increasing when $E[R(\mathbf{x},\mathbf{r})]$ increases and $V[R(\mathbf{x},\mathbf{r})]$ is held fixed, and increasing when $V[R(\mathbf{x},\mathbf{r})]$ decreases and $E[R(\mathbf{x},\mathbf{r})]$ is held fixed, all potentially maximizing solutions of the substitute problem (2) can be obtained by solving

$$\begin{aligned} &\max\{E[R(\mathbf{x},\mathbf{r})]\} \\ &\min\{V[R(\mathbf{x},\mathbf{r})]\} \\ &s.t. \quad \mathbf{x} \in S \end{aligned} \tag{8}$$

for all $\mathbf{x}$-vectors in S from which it is not possible to increase expected portfolio return without increasing portfolio variance, or decrease portfolio variance without decreasing expected portfolio return. Note that (8) is equivalent to the problem posed by Markowitz [34], but justified from an explicitly stochastic programming point of view.

Since it is also assumed that the r_j in $R(\mathbf{x},\mathbf{r})$ are from distributions whose means μ_j, variances σ_{jj} and covariances σ_{ij} are known, it is interesting to see how stochastic programming problem (1) under the above rationale is operationalized in the form of a problem with two *deterministic* objectives. With S from (1), this is seen more clearly by re-expressing (8) as

$$\max\{\sum_{i=1}^{n}\mu_j x_j\} \tag{9}$$
$$\min\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}x_j\}$$
$$s.t. \quad (x_1,\ldots,x_n)\in S=\{\mathbf{x}\in R^n \mid \sum_{i=1}^{n} x_i=1,\ \alpha_i\le x_i\le\omega_i\}$$

Standard theory embraces a second set of assumptions that through (2), also leads to (8), and hence (9). If we drop the condition that our risk-averse U is quadratic, but add the condition that all security returns $(r_1,\ldots,r_n)$ are multivariate normally distributed, then all potentially maximizing solutions of this U in (2) are obtained by solving (8), or equivalently (9), in the same way.

Now, with all of the images of the **x**-vectors that solve (8), or equivalently (9), tracing a positively-sloped curve in (standard-deviation, expected-return) space, it is assumed that an investor is able to identify the point on the curve that maximizes his substitute problem (2), thus specifying for the investor, via an inverse image of the point, an optimal (portfolio) solution for (1). It is in this way, under the acceptance of the rationale of this section, that (9) has emerged as the reigning operationalization for solving portfolio selection problems.

3 Suitable-Portfolio Theory

With standard theory as a reference point, let us now introduce theory with regard to suitable-portfolio investors. Actually, it is from the very start that the theory of a suitable-portfolio investor deviates from that of a standard investor. Whereas a standard investor starts with the need to solve the single-objective formulation (1), a suitable-portfolio investor starts with the need to solve the multiple objective formulation

$$
\begin{aligned}
&\max\{\sum_{j=1}^{n} r_j x_j = R(\mathbf{x},\mathbf{r})\} \qquad (10)\\
&\max\{\sum_{j=1}^{n} c'_{2j} x_j = g'_2(\mathbf{x})\}\\
&\qquad\vdots\\
&\max\{\sum_{j=1}^{n} c'_{pj} x_j = g'_p(\mathbf{x})\}\\
&\quad s.t. \quad \mathbf{x} \in S
\end{aligned}
$$

in which the extra objectives are $p-1$ additional criteria to be respected by the investor, and the primes on the c_{ij} and g_i indicate that all of the additional objectives are written in maximization form.

With the first objective remaining stochastic, we assume that all of the additional objectives 2 to p can be acceptably modelled deterministically. There is no problem with objectives such as the number of securities in a portfolio, the maximum amount invested in any security, or the amount of short selling involved, because these objectives are deterministic in the first place. While there could be a problem with objectives like dividends, social responsibility, and R&D, it is our position in this paper that concerns about variability in the outcomes of these objectives are of a different order of magnitude than with portfolio return, and can in many if not most practical cases be safely ignored.

As for interpreting what it means to solve (10) with first-objective stochastic, let $U\colon M \to R$ now be the investor's utility function in p arguments (one for each objective). With D as defined earlier, it suffices for the domain of U to be given by $M = D \times R^{p-1}$. As for properties, let U be coordinatewise strictly increasing in all p arguments, and furthermore, be strictly concave and increasing (risk averse) in the first argument.

Under the interpretation that it is also acceptable here to proceed along the lines of expected utility maximization, it is then considered that any $\mathbf{x}$ that solves the substitute problem

$$
\begin{aligned}
&\max\{E[\,U(R(\mathbf{x},\mathbf{r}), g'_2(\mathbf{x}), \ldots, g'_p(\mathbf{x}))]\} \qquad (11)\\
&\quad s.t. \quad \mathbf{x} \in S
\end{aligned}
$$

is an optimal portfolio solution for any investor whose formulation is as specified in (10). As for operationalizing (11), guidance is drawn from Theorem 1.

Theorem 1 *Let*

- **(a)** $R(\mathbf{x},\mathbf{r})$ *be stochastic as described in* (1)
- **(b)** *the* r_j *in* $R(\mathbf{x},\mathbf{r})$ *be from distributions whose means* μ_j*, variances* σ_{jj} *and covariances* σ_{ij} *are known*

(c) $g'_2(\mathbf{x}), \ldots, g'_p(\mathbf{x})$ *be deterministic functions as defined in* (10)
(d) U *be defined as above.*

Then if

(i) U *is also quadratic over* M *in the first argument and all security returns* $(r_1, \ldots, r_n)$ *are continuously distributed, or*
(ii) *the security returns* $(r_1, \ldots, r_n)$ *are multivariate normally distributed*

all potentially maximizing solutions of (11) *can be obtained by solving*

$$\begin{gathered} \max\{\sum_{i=1}^{n} \mu_j x_j\} \\ \min\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j\} \\ \max\{g'_2(\mathbf{x})\} \\ \vdots \\ \max\{g'_p(\mathbf{x})\} \\ s.t. \quad \mathbf{x} \in S \end{gathered} \tag{12}$$

for all $\mathbf{x}$*-vectors of* (12) *from which it is not possible to increase one of the maximizing objectives without decreasing at least one of the other maximization objectives or increasing the minimization objective, or decrease the minimization objective without decreasing at least one of the maximization objectives.*

The proof of Theorem 1 can be found in the Appendix. Unfortunately, computing all such $\mathbf{x}$-vectors of (12) when $p \geq 2$ is more difficult than when extra objectives do not exist because they cannot normally be computed by the methods of single-objective or parametric mathematical programming. Hence their computation is left to the topic of multiple criteria optimization. Moreover, the images of the $\mathbf{x}$-vectors in (12) are vectors of dimensionality $p+1$ so that when there are extra objectives they do not lend themselves to graphical representation as in the former case. To address the difficulties inherent in (12), we now overview multiple criteria optimization.

4 Multiple Criteria Optimization

In deterministic operations research, there is the multiple criteria optimization problem that is not normally addressed by the methods of traditional mathematical programming. While on first appearance, other than for possessing a stack of deterministic objectives instead of just one, it does not look much different from an ordinary mathematical programming problem, but its solution is much more involved. Facilitating both maximization and minimization objectives, a multiple criteria optimization problem can be expressed as

$$\begin{aligned} \max \textit{ or } \min \{f_1(\mathbf{x}) &= z_1\} \\ &\vdots \\ \max \textit{ or } \min \{f_k(\mathbf{x}) &= z_k\} \\ \text{s.t.} \qquad \mathbf{x} &\in S \end{aligned} \tag{13}$$

in which the f_i are deterministic functions, $\mathbf{x} \in R^n$, k is the number of objectives, and the z_i are *criterion values*. In single-objective programming there is only the feasible region S in *decision space* R^n. But in multiple objective programming there is the feasible region $Z = \{\mathbf{z} \mid \mathbf{z} = f(\mathbf{x}),\ \mathbf{x} \in S\}$ in *criterion* (or evaluation) *space* R^k. In this way under the f_i, each $\mathbf{x} \in S$ in decision space has an *image* $\mathbf{z} \in Z$ in criterion space, and each $\mathbf{z} \in Z$ in criterion space has at least one *inverse image* $\mathbf{x} \in S$ in decision space.

Criterion vectors $\mathbf{z} \in Z$ are either *nondominated* or *dominated*. Let $J^+ = \{\, i \mid f_i(\mathbf{x}) \text{ is to be maximized}\}$ and $J^- = \{j \mid f_j(\mathbf{x}) \text{ is to be minimized}\}$. Then we have

Definition 1 *Let* $\bar{\mathbf{z}} \in Z$. *Then* $\bar{\mathbf{z}}$ *is* nondominated *in* (13) *if and only if there does not exist another* $\mathbf{z} \in Z$ *such that* **(i)** $z_i \geq \bar{z}_i$ *for all* $i \in J^+$, *and* $z_j \leq \bar{z}_j$ *for all* $j \in J^-$, *and* **(ii)** $z_i > \bar{z}_i$ *or* $z_j < \bar{z}_j$ *for at least one* $i \in J^+$ *or* $j \in J^-$. *Otherwise,* $\bar{\mathbf{z}} \in Z$ *is* dominated.

The set of all nondominated criterion vectors is an important set, is designated N, and is called the *nondominated set*. In conjunction with vectors $\mathbf{z} \in Z$ being either nondominated or dominated, points $\mathbf{x} \in S$ in decision space are either *efficient* or *inefficient* as follows.

Definition 2 *Let* $\bar{\mathbf{x}} \in S$. *Then* $\bar{\mathbf{x}}$ *is* efficient *in* (13) *if and only if its image criterion vector* $\bar{\mathbf{z}} = (f_1(\bar{\mathbf{x}}), \ldots, f_k(\bar{\mathbf{x}}))$ *is nondominated, that is, if and only if* $\bar{\mathbf{z}} \in N$. *Otherwise,* $\bar{\mathbf{x}}$ *is* inefficient.

The set of all efficient points is designated E and is called the *efficient set*. While nondominance is a criterion space concept, we note that in multiple criteria optimization, efficiency is only a decision space concept.

To define optimality in multiple criteria optimization, let $U\colon Z \to R$ be the decision maker's k-argument utility function where the k arguments are the z_i. Then, any $\mathbf{z}^{\text{o}} \in Z$ that maximizes U over Z is an *optimal criterion vector*, and any inverse image of $\mathbf{z}^{\text{o}}$, that is, any $\mathbf{x}^{\text{o}} \in S$ such that $(f_1(\mathbf{x}^{\text{o}}), \ldots, f_k(\mathbf{x}^{\text{o}})) = \mathbf{z}^{\text{o}}$, is an *optimal solution*. We are interested in the efficient and nondominated sets because if U is such that *more-is-always-better-than-less* for each z_i, $i \in J^+$, and *less-is-always-better-than-more* for each z_j, $j \in J^-$, then any optimal criterion vector is a member of the nondominated set N, and any inverse image of an optimal criterion vector is a member of the efficient set E. This means that to find an optimal criterion vector, one only needs to find the most preferred criterion vector in N.

To obtain an optimal solution, we proceed as follows. Search among the criterion vectors of all $\mathbf{x}$-vectors that are members of E to identify the best criterion vector in N. Despite the fact that N is always some portion of the surface of Z, finding a most preferred criterion vector is often a non-trivial task because of the size or eccentricity of N. Nevertheless, assuming that a most preferred (i.e., optimal) criterion vector $\mathbf{z}^{\text{o}}$ can eventually be found, it is only necessary to obtain an $\mathbf{x}^{\text{o}}$ inverse image to know what to implement.

In another way of looking at nondominated criterion vectors, they are all *contenders for optimality.* By this it is meant that for each nondominated criterion vector, there exists a utility function U, with the property that more-is-always-better-than-less in the argument of each objective that is to be maximized and less-is-always-better-than-more in the argument of each objective that is to be minimized, that makes it uniquely optimal. As for dominated criterion vectors, they are all *noncontenders* for optimality because for them no such utility functions exist.

While the field of multiple criteria optimization has concerns about the full computation or characterization of efficient and nondominated sets, this is normally of help only in small problems. However, for addressing larger problems with objectives in the range of 3 to about 7, the field of multiple criteria optimization is mostly concerned with interactive procedures for repetitively sampling in a progressively more concentrated fashion the nondominated set N until a *final solution* is obtained. In practice, a final solution is either an optimal solution, or a solution close enough to being optimal to terminate the decision process.

To reconcile the terminology of Sections 2 and 3 with the terminology of multiple criteria optimization of this section, "all potentially maximizing solutions" of substitute problems (2) and (11), or alternately "all $\mathbf{x}$-vectors solutions" of (9) and (12), constitute efficient sets, and full sets of "images" constitute nondominated sets.

While the operationalization (9) of the portfolio problem of standard theory (1) does not need the methods of multiple criteria optimization (it can be addressed by already existing methods), the operationalization

$$\begin{aligned}
\max\{\sum_{i=1}^{n}\mu_j x_j = z_1\} \qquad (14)\\
\min\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}x_j = z_2\}\\
\max\ or\ \min\{\sum_{j=1}^{n} c_{2j}x_j = z_3\}\\
\vdots\\
\max\ or\ \min\{\sum_{j=1}^{n} c_{pj}x_j = z_k\}\\
s.t.\quad (x_1,\ldots,x_n)\in S=\{\mathbf{x}\in R^n \mid \sum_{i=1}^{n} x_i = 1,\ \alpha_i \le x_i \le \omega_i\}
\end{aligned}$$

of the portfolio problem of suitable-portfolio theory (10) does need the methods of multiple criteria optimization. The reason is that the mathematics of problem solving changes as we transit from problems with two objectives to problems with three or more objectives. Since none of the methods for solving (two-objective) mean-variance formulations, such as Markowitz's critical line method [37] or e-constraint approaches (discussed in Section 7), can easily be extended to computing all of E and N in problems with additional objectives, this is where multiple criteria optimization comes in as its specialty is problems with three or more objectives.

5 Standard Model Variants

When looking through the portfolio chapters of typical university investments texts (Bodie, Kane and Marcus [8], for example), it is hard to miss seeing graphs of bullet-shaped regions. Upon closer examination, with standard deviation on the horizontal and expected return on the vertical, these graphs are recognized as Z feasible regions in criterion space of implementations of the standard model's operationalization (9).

Because standard deviation is to be minimized and expected return is to be maximized, the nondominated set is to the "northwest." In this way, the nondominated set is the upper portion (the portion that is positively sloped) of the minimum standard deviation boundary of a bullet-shaped Z. In finance, this upper portion of the minimum standard deviation (leftmost) boundary is called the "efficient frontier." However, this causes a terminological conflict with the distinction drawn earlier about efficiency being a decision space concept and nondominance being a criterion space concept. Rather than efficient frontier, we will call it the "nondominated frontier," not only because this is consistent with the terminology of multiple criteria optimization, but because nondominated is a more intuitive term for use in criterion space.

With regard to the standard model's operationalization (9), there are two variants. One is the *unrestricted-variable* model in which all lower bounds are $-\infty$ and all upper bounds are $+\infty$. This causes this model variant's bullet-shaped Z to be unbounded to the right. This model is often the favorite in teaching and academic research because of its nice mathematical properties when $\mathbf{V}$, the $n \times n$ covariance matrix of the σ_{ij}, is nonsingular. This then causes the model's minimum standard deviation boundary of its Z to be a hyperbola, which when exploited, enables virtually any piece of information about the unrestricted-variable model to be obtainable in closed-form (see for example Roll [42], pp. 158-165).

Starting with a vector of individual security mean returns and a nonsingular matrix of covariances of returns

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & & \\ \vdots & & & \vdots \\ \sigma_{n1} & & \cdots & \sigma_{nn} \end{bmatrix}$$

we demonstrate as follows. First form

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\mu}^T\mathbf{V}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}^T\mathbf{V}^{-1}\mathbf{1} \\ \mathbf{1}^T\mathbf{V}^{-1}\boldsymbol{\mu} & \mathbf{1}^T\mathbf{V}^{-1}\mathbf{1} \end{bmatrix} \equiv \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{15}$$

in which $\mathbf{1} \in R^n$ is a vector of ones because $\mathbf{A}$ is used in many of the formulas. For example, (a) the expected return of the portfolio with minimum standard deviation is given by

$$\mu_p^{min} = \frac{b}{c}$$

(b) the standard deviation $\sigma_p(\mu_p)$ of a point on the nondominated frontier whose expected return is $\mu_p \geq \mu_p^{min}$ is given by

$$\sigma_p(\mu_p) = \sqrt{[\mu_p \;\; 1]\,\mathbf{A}^{-1}\begin{bmatrix} \mu_p \\ 1 \end{bmatrix}}$$

and (c) the investment proportion vector $\mathbf{x}^p \in E$ whose image is the nondominated criterion vector $(\sigma_p(\mu_p), \mu_p)$ is given by

$$\mathbf{x}^p = \mathbf{V}^{-1}\,[\boldsymbol{\mu} \;\; \mathbf{1}]\,\mathbf{A}^{-1}\begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

and so forth. For checking dimensionalities, $\mathbf{A}$ is 2×2, the transpose of $[\mu_p \;\; 1]$ is 2×1, and $[\boldsymbol{\mu} \;\; \mathbf{1}]$ is $n \times 2$. The supposed advantages of the unrestricted-variable model are that it enables one to focus better on the relationships among variables and to teach portfolio theory without necessitating the inclusion of mathematical programming in the curriculum (which it hardly ever is any more in finance).

The other variant of operationalization (9) is the *restricted-variable* model in which at least some of the lower and upper bounds are finite. As a consequence, (a) bullet-shaped feasible region Z is only smaller, (b) the minimum standard deviation boundary, instead of being a single hyperbola, is piecewise hyperbolic, (c) closed-form results are in general no longer possible, and (d) mathematical programming is now required for computing the nondominated frontier.

6 Portfolio Theory Assumptions Examined

Because of the importance of portfolio selection to the branch of finance known as "modern portfolio analysis" for the past four decades, let us study its assumptions. Given that the world is not the same as it was in the 1960s when the assumptions were set in place, that multiple criteria optimization did not exist at the time, and that there has never been much academic interest in individual investors, the assumptions may well be in need of an overhaul. Similar to the list in Elton and Gruber [17], the assumptions, classified here into the four groups of overall environment, preferences, expectations and model convenience, are as follows.

- **(a)** There are many investors, each small, none of which can affect prices.
- **(b)** There are no taxes.
- **(c)** There are no transactions costs.
- **(d)** Each investor's asset universe is all publicly-traded securities.

- **(e)** All investors are rational mean-variance optimizers.
- **(f)** All investors have expected utility functions whose contours are strictly convex-to-the-origin.

- **(g)** All investors share the same expected returns, predicted variances, and predicted covariances about the future. This is called *homogeneous expectations.*

- **(h)** Each security is infinitely divisible.
- **(i)** Unlimited short selling is allowed.
- **(j)** All investors have the same single holding period.

As one can see, the effect of the assumptions is to standardize investors to the point that they all have the same single goal of maximizing portfolio return, have the same feasible region, and have the same nondominated frontier. In fact, the only latitude given to an investor under these assumptions is in the selection of a final point from the curvilinear nondominated frontier they all share in common. Clearly the philosophy of these assumptions is at odds with the concept of a suitable-portfolio investor. Let us now look at the assumptions in greater detail.

In the overall environment group, we have the first four assumptions. Assumption (a) is akin to the perfect competition assumption in economics, and assumptions (b) and (c) are included to avoid the artificialities of tax consequences and the complexities of transaction costs. These assumptions are not criticized because their lack of realism is viewed as being outweighed by their usefulness in being able to establish benchmarks against which the costs of the real world's imperfections can be measured. Assumption (d) is a little different. With tens of thousands of publicly-traded securities, it is impossible to imagine how anyone could have all publicly-traded securities as their asset universe. Clearly, this assumption is for standardization, but it is of concern to us because the size of one's asset universe affects the size of one's feasible region, and hence the location of one's nondominated frontier.

In the preferences group, we first have assumption (e) which states that it suffices for investors to consider only expected return and standard deviation of return to evaluate portfolios. This is too restrictive as regards suitable-portfolio investors, and will later be relaxed in our efforts to develop a more inclusive and broad-based theory of portfolio selection. As for assumption (f), strictly convex-to-the-origin contours mean that, in conjunction with convex Zs, an investor's optimal criterion vector is always a singleton point that can be found somewhere on the nondominated frontier. Strictly convex-to-the-origin contours also is a common assumption in multiple criteria optimization, so there appears to be no pressing need to deviate from it in this paper.

In the expectations group, there is only assumption (g) which states that all investors are the same as regards their forecasts of the future. This is clearly not the case. In fact, it is almost the hallmark of investors to specialize in different prognostications. This is important because if everyone has different forecasts, everyone will have different feasible regions and hence different nondominated frontiers. Once again, we have another case in which the attempt to standardize does not appear to be defensible.

In the model convenience group, we have the last three assumptions. These assumptions appear to have outlived their usefulness. With faster computers, improved optimization software, and advances in evolutionary and local search algorithms (see Deb [13] and Gandibleux, Caballero and Molina [19]), progress is possible on integrating almost any type of variable condition (for instance, integer and semi-continuous variables) into portfolio problems of meaningful size. With regard to assumption (i), unlimited short selling is a stretch of the imagination and to allow it creates an unrealistically large feasible regions. For most investors, short selling consumes cash, and a limit is quickly reached. Even hedge funds soon hit limits on short selling. While moderate amounts of short selling are reasonable, unlimited amounts clearly are not. As for assumption (j) which states that all investors have the same holding period, this is yet another attempt to standardize that cannot be sustained.

7 Sensitivities of the Nondominated Frontier

In this section we consider sensitivity issues in the (bicriterion) mean-variance model (9) with regard to reasonable parameters and deviations from the assumptions of modern portfolio analysis of the previous section. We do this by noting the changes undergone by the nondominated frontier as the parameters in question change.

First, we look at the sensitivity of the nondominated frontier to changes in a common upper bound on all of the investment proportion weights. Following this, we discuss the likely sensitivity of the nondominated frontier to changes in other parameters such as a portfolio dividend requirement, a portfolio social responsibility quotient, and other matters that may be of concern.

Because of the difficulty in locating good parametric quadratic programming software, the optimization runs for the experiments of this section were carried out in accordance with the following 5-step procedure. The purpose is to compute enough points so that by connecting them the nondominated frontier in question can be accurately drawn.

1. Convert the expected portfolio return objective of (9) to a $\geq$ constraint with right-hand side ρ to form what is recognized in multiple criteria optimization as an e-constraint program.
2. Add any additional constraints to S required by the experiment being conducted.
3. Make sure the parameter is set to its appropriate value.
4. Successively solve the e-constraint program for a sequence of, say 30, different right-hand side ρ values. From the solution of each e-constraint run, extract standard deviation value $\sigma[\,\rho\,]$ to form nondominated point $(\sigma[\,\rho\,], \rho)$.
5. Connect the nondominated points resulting from the 30 runs with a smooth line to form the computed nondominated frontier. Go to Step 3.

To illustrate, let us track through the process by which the nondominated frontiers of Figure 1, as a function of a common upper bound $\bar{\omega}$ on the x_j, are computed for display in an instance when $n = 30$. Following the 5-step procedure, we form the e-constraint program of Step 1; skip Step 2; and set all α_j to 0 (our choice) and install our first value for $\bar{\omega}$ (we set $\bar{\omega} = 1.00, 0.10, 0.06$ in our analysis) in Step 3 to form

$$\min \{ \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j \} \tag{16}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \mu_j x_j \geq \rho$$

$$(x_1, \ldots, x_n) \in S = \{ \mathbf{x} \in R^n \mid \sum_{j=1}^{n} x_j = 1,\ 0 \leq x_j \leq \bar{\omega} \}$$

Cycling through the 5-step procedure once for each $\bar{\omega}$ value, solving 30 e-constraint programs each time, we have the three nondominated frontiers of Figure 1. The top frontier is from $\bar{\omega} = 1.00$, the middle frontier is from $\bar{\omega} = 0.10$, and the short bottom frontier is from $\bar{\omega} = 0.06$.

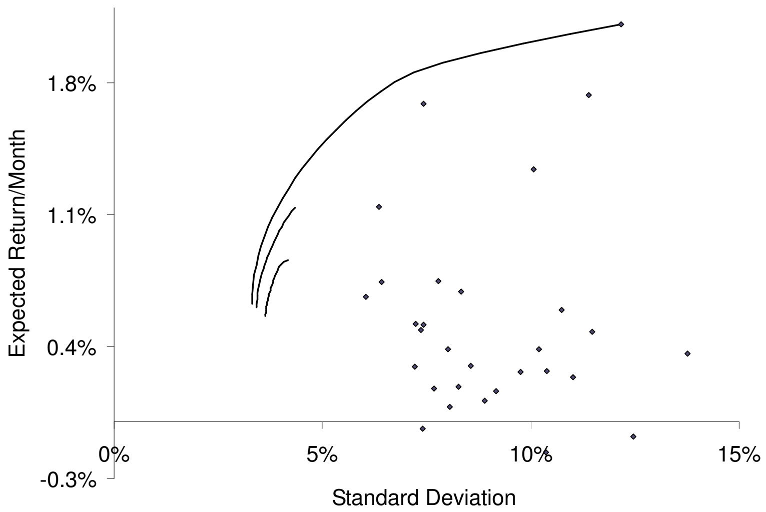

Figure 1: Nondominated frontiers with upper bound parameter $\bar{\omega} = 1.00, 0.10, 0.06$, all $\alpha_j = 0$, and $n = 30$.

As seen in Figure 1, the nondominated frontier undergoes major changes as we move through different values of $\bar{\omega}$. Let the risk-free rate be 0.4%/month. If our interest is in the point that maximizes the Sharpe ratio (tangent point on the nondominated frontier to a line drawn from 0.4% on the vertical axis), then we would be about 50% up the $\bar{\omega} = 1.00$ nondominated frontier, at the top of the $\bar{\omega} = 0.10$ nondominated frontier, and nearly at the top of the $\bar{\omega} = 0.06$ nondominated frontier. These are three notably different points, and this represents considerable sensitivity. This would not be so serious if an investor were in a position to know *a priori* his or her most appropriate $\bar{\omega}$. But with all of the tradeoffs among standard deviation, expected return, and perhaps other criteria, it is almost certainly not possible to know an optimal value $\bar{\omega}$ in advance. This is rationale as to why, if $\bar{\omega}$ is of concern to an investor, $\bar{\omega}$ should be modelled as an objective rather than a constraint.

To consider another experiment (results not shown) to examine the sensitivity of the nondominated frontier to changes in a lower bound $\bar{\delta}$ on portfolio dividends, we would again utilize the 5-step procedure but with

$$\min\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}x_j\} \tag{17}$$
$$\text{s.t.} \quad \sum_{j=1}^{n}\mu_j x_j \geq \rho$$
$$\sum_{j=1}^{n} d_j x_j \geq \bar{\delta}$$
$$(x_1,\ldots,x_n) \in S = \{\mathbf{x} \in R^n \mid \sum_{j=1}^{n} x_j = 1,\ \alpha_j \leq x_j \leq \omega_j\}$$

(for some values of α_j and ω_j) for different values of parameter $\bar{\delta}$ (such as, $\bar{\delta} =$ .01, .02, .03, .04).

Although the results of the experiment are not given here, they would show, as in Figure 1, similar sensitivities. If we continued on with experiments to test the sensitivity of the nondominated frontier to changes in social responsibility aspirations, short selling limitations, the number of securities in a portfolio, and so forth, we would see more of the same, providing a quantitative argument for pursuing a multiple criteria approach.

8 Expanded Multiple Criteria Formulation

In multiple criteria there is always a line that must be drawn between what is most appropriately modeled as an objective and what is most appropriately modeled as a constraint, but in finance, with mean-variance optimizer assumption (e), the line has most likely been drawn too soon. Generally, in multiple criteria, we recognize a constraint from an objective as follows. If, when modeling a constraint, we realize that we can not easily fix a right-hand side value without knowing the levels of the other criteria, then we are probably looking at an objective. In fact, this is the case with upper bound $\bar{\omega}$ and, depending on the investor, can easily be the case with dividends, social responsibility, number of securities in the portfolio, short selling, and other concerns. Consequently, beyond just risk and return, an investor, for example, could easily be looking at an expanded multiple criteria optimization formulation such as

$$
\begin{aligned}
\min \{f_1(\mathbf{x}) &= \text{risk}\} \\
\max \{f_2(\mathbf{x}) &= \text{return}\} \\
\max \{f_3(\mathbf{x}) &= \text{dividends}\} \\
\min \{f_4(\mathbf{x}) &= \text{maximum investment proportion weight}\} \\
\max \{f_5(\mathbf{x}) &= \text{social responsibility}\} \\
\min \{f_6(\mathbf{x}) &= \text{number of securities in portfolio}\} \\
\min \{f_7(\mathbf{x}) &= \text{short selling}\} \\
\text{s.t.} \qquad & \mathbf{x} \in S
\end{aligned} \tag{18}
$$

Note that there is nothing sacrosanct about the above number or mix of objectives that must be present in a given investor's expanded formulation. Other objectives that could well have been used in place of, or in addition to, those listed in (18) might involve

$$
\begin{aligned}
\max \{f_8(\mathbf{x}) &= \text{liquidity (see Lo, Petrovz and Wierzbicki [31])}\} \\
\min \{f_9(\mathbf{x}) &= \text{turnover}\} \\
\max \{f_{10}(\mathbf{x}) &= \text{skewness}\} \\
\max \{f_{11}(\mathbf{x}) &= \text{amount invested in R\&D (see Guerard and Mark [21])}\} \\
\min \{f_{12}(\mathbf{x}) &= \text{tracking error}\}
\end{aligned}
$$

Criteria beyond risk and return are not unreasonable. As for minimizing the number of securities in a portfolio, each additional security is extra administration, and beyond a point, can be a managerial distraction. As for short selling, this may be undesirable, but might be tolerated if worth it in terms of what can be achieved on the other criteria. The point here is that we have no way of knowing an optimal value of a criterion without knowing the values of the other criteria, and the smallest set containing all necessary information for us to make a decision is the nondominated set.

Recognizing that investors may possess multiple criteria, that investors may not possess the same criteria, and that investors may hold different forecasts of the future, let us consider a revision of the ten assumptions of Section 6 for non-standard investors as follows.

- **(a)** There are many investors, each small, none of which can affect prices (same).
- **(b)** There are no taxes (same).
- **(c)** There is no longer any need to assume no transaction costs (because transaction costs and other related nuances can be handled through liquidity and turnover criteria if so desired).
- **(d)** An investor's asset universe can be any subset of securities (for most investors, this means usually not more than a few hundred).

(e) Investors may possess any mix of up to about seven objectives (limited by the fact that multiple criteria decision making is not yet well developed for problems with more than this number of objectives).

(f) All investors have expected utility functions whose contours are strictly convex-to-the-origin (same).

(g) Heterogeneity of expectations. That is, investors can be expected to hold differing forecasts about any investment variable (individualism is not only allowed on expectations, it is to be expected).

(h) Each security is infinitely divisible (with faster computers and improved optimization software, this assumption should be dropped, but this is another whole topic).

(i) Short selling is only allowed under realistic restrictions (unrealistic and unlimited short selling ruled out).

(j) An investor may have any holding period (individualism allowed on holding periods, too).

9 Nondominated Surfaces

Note that nothing has been done to standard theory finance. We have only experimented with adding criterion space dimensions to it for suitable-portfolio investors. In addition to the nondominated frontier sensitivity seen in Section 7, might there be any other evidence to corroborate our multiple criteria hypothesis? Consider the "market portfolio," the portfolio containing all securities in proportion to their market capitalization. By theory, the market portfolio is on the nondominated frontier and is an investor's optimal portfolio of risky assets. However, since the market portfolio is impractical, broad-based indices such as the S&P 500 or Wilshire 5000 are used as surrogates. But empirically, the surrogates plot disconcertingly deep below the nondominated frontier, a fact that standard theory has difficulty explaining.

If the criteria in (18) indeed trade-off against one another, then the nondominated set is no longer a "frontier" as before, but is now a portion of the *surface* of the investor's k-dimensional Z. Let us conduct a little experiment. Assume that an investor's optimal portfolio is in the middle of his or her nondominated set. While the "buried" nature of the market portfolio is an anomaly in standard theory, in suitable-portfolio theory, buried is exactly what we would expect.

To illustrate, let us assume that the Z of a standard investor is the region bounded by the ellipse in Figure 2. Here, the nondominated set is the portion of the boundary of the elliptical Z as indicated. Let us further assume that the Z of a suitable-portfolio investor is the region bounded by a k-dimensional *ellipsoid* with the same major and minor axes in (risk, return) space. Here, the nondominated set is the portion of the surface of the ellipsoidal Z that projects onto the upper left quarter of the 2-dimensional elliptical feasible region. If the optimal portfolio is in the middle of the nondominated set like $\mathbf{z}^2$ is for a standard investor, then the optimal portfolio would be at $\mathbf{z}^3$ for a suitable-

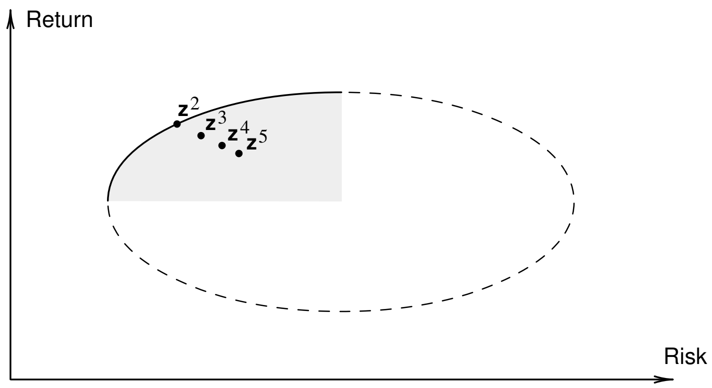

Figure 2: Ellipsoidal feasible regions projected onto two-dimensional (risk, return) plane

portfolio investor with a 3-dimensional ellipsoidal Z, at $\mathbf{z}^4$ for a suitable-portfolio investor with a 4-dimensional ellipsoidal Z, and so forth, becoming deeper and deeper as the number of criteria increases. Thus, being buried in (risk, return) space is no anomaly in multiple criteria portfolio selection.

10 Computational Solvability

While not the thrust of this paper as it is a whole additional research area, a few comments are made about the solvability of expanded multiple criteria portfolio selection optimization formulations as in (18) to demonstrate the computational viability of what is proposed in this paper. By "solvability" we are referring to the two-stage process of, first, computing or characterizing the nondominated surface, and then, searching the obtained representation for the most satisfactory point.

With regard to Stage 1 possibilities, let a given investor's expanded formulation have one quadratic and $k-1$ linear objectives as in

$$\begin{aligned} &\min \{\mathbf{x}^T \mathbf{Q} \mathbf{x}\} \\ &\max \{(\boldsymbol{\mu}^1)^T \mathbf{x}\} \\ &\quad \vdots \\ &\max \{(\boldsymbol{\mu}^{k-1})^T \mathbf{x}\} \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{19}$$

in which S is convex and $\mathbf{Q} \in R^{n \times n}$ is positive semidefinite[1]. Now let

$$\Lambda = \{\boldsymbol{\lambda} \in R^k \mid \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1\}.$$

With $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda$, form

$$\begin{array}{ll} \max \{ -\lambda_1(\mathbf{x}^T\mathbf{Q}\mathbf{x}) + \lambda_2(\boldsymbol{\mu}^1)^T\mathbf{x} + \cdots + \lambda_k(\boldsymbol{\mu}^{k-1})^T\mathbf{x}\} \\ \text{s.t.} \qquad \mathbf{x} \in S \end{array} \tag{20}$$

whose solution produces a point on the nondominated surface of (19). The power of (20) is that a point is on the nondominated surface if and only if it can be generated in this way for some $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda$. Since (20) is only a quadratic programming problem, it can be readily solved by software such as provided by Matlab [38] and other vendors. Then, by repetitively solving (20) for different vectors $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda$, a dispersion of points characterizing the nondominated surface can be obtained. With the QUADPROG procedure of Matlab being able to solve (20) in less than about 15 seconds[2] for problems with up to about 200 variables, up the several thousand points representing a nondominated surface can be generated overnight. Clearly, this is a brute force approach, but more sophisticated approaches are on the horizon.

In current research by the authors, it has been learned that the nondominated surface of problems such as (19) consists of numerous parabaloidic platelets (like the tiles on the front of the space shuttle). Although not yet fully described in manuscript form, an algorithm has been developed and coded for computing the corner points in R^n of all such platelets. Then using the corner points, and convex combinations of the corner points, a nondominated surface can be represented discretely to any degree of resolution. Preliminary results indicate that nondominated surfaces can be discretely characterized in this way in only a fraction of the time required by the weighted-sums approach of (20).

Even if some of the objectives or constraints of (19) are more severely nonlinear or possess non-smooth aspects, advances from the field of multicriterion evolutionary algorithms (see Deb [13]) can probably be counted on in the near future as that field become more decidedly focused on the discretization of surfaces.

As for the Stage 2 task of searching discretized nondominated surfaces, we look to the methods of interactive multiple objective programming. Selecting from these methods, we see three strategies adaptable to the needs of expanded portfolio selection. One is to iteratively sample the set of discrete points in a progressively more concentrated fashion as in the Tchebycheff Method of Steuer and Choo [44]. Another is to conduct successive projected line searches across the discretized nondominated surface as in Korhonen and Karaivanova [30], which can be likened to driving an SUV across the bumpy surface of the moon. A third involves the terative classification of the criteria into those to be improved,

[1] All covariances matrices are positive semidefinite.

[2] on a 3.06GHz Dell Pentium 4 desktop

those to be held at roughly the same level, and those that are allowed to be deteriorated, as in the NIMBUS approach of Miettinen [40].

With rudiments of Stage 1 and Stage 2 procedures already in place, there is no reason to believe that more advanced capabilities for solving multiple criteria portfolio optimization problems won't be produced in the future.

11 Concluding Remarks

Since the success of any economy is in large part related to the effectiveness by which the economy's society allocates its scare capital resources, no stone should be left unturned in trying to better understand the investment process. This includes theory, education, and practice. Although this paper pursues an unusual approach, we are not alone, as from Section 1 and the list of references, there are others with similar emerging views.

With the assumptions of Section 8 subsuming the assumptions of Section 6, operationalization (14) subsuming operationalization (9), our discussions about nondominated frontier sensitivity, and the buried nature of the market portfolio, it is not an unreasonable proposition that the "modern portfolio theory" of today may best be viewed as a projection onto the risk-return plane of the real multiple criteria problem from higher dimensional space.

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Appendix: Proof of Theorem 1

Choose $\mathbf{x} \in S$ maximizing expected utility and define the corresponding stochastic objective vector by

$$\mathbf{y}(\mathbf{r}) := (R(\mathbf{x}, \mathbf{r}), g_2'(\mathbf{x}), \dots, g_p'(\mathbf{x}))^T \ .$$

We assume that $\mathbf{x}$ is not efficient for the substitute problem, i.e., there exists an $\bar{\mathbf{x}} \in S$ such that the substitute problem's objective values $\bar{\mu}, \bar{\sigma}^2, \bar{y}_2, \dots, \bar{y}_p$ of $\bar{\mathbf{x}}$ dominate the objective values $\mu, \sigma^2, y_2, \dots, y_p$ of $\mathbf{x}$ such that

$$(\bar{\mu}, -\bar{\sigma}^2, \bar{y}_2, \dots, \bar{y}_p)^T \geq (\mu, -\sigma^2, y_2, \dots, y_p)^T$$

and ">" for some index $i \in \{0, \dots, p\}$, where the index of μ is 0, the index of σ^2 is 1, the index of y_2 is 2, etc. We will show that

$$E[\,U(\bar{y}_1(\mathbf{r}), \bar{y}_2, \dots, \bar{y}_p)\,] \overset{(1)}{\geq} E[\,U(\bar{y}_1(\mathbf{r}), y_2, \dots, y_p)\,] \overset{(2)}{\geq} E[\,U(y_1(\mathbf{r}), y_2, \dots, y_p)\,]$$

where (1) is ">" if $\mathbf{x}$ is dominated on the (expected-value, variance) part $i \in \{0, 1\}$, and (2) is ">" if $\mathbf{x}$ is dominated on the deterministic part, i.e. if $i \in \{2, \dots, p\}$. This yields the contradiction

$$E[\,U(\bar{\mathbf{y}}(\mathbf{r}))\,] = E[\,U(\bar{y}_1(\mathbf{r}), \bar{y}_2, \dots, \bar{y}_p)\,] > E[\,U(y_1(\mathbf{r}), y_2, \dots, y_p)\,] = E[\,U(\mathbf{y}(\mathbf{r}))\,]$$

Part (1): Since U is coordinatewise strictly increasing in $y_2, \dots, y_p$

$$U(\bar{y}_1(\mathbf{r}), \bar{y}_2, \dots, \bar{y}_p) \ \geq \ U(\bar{y}_1(\mathbf{r}), y_2, \dots, y_p)$$

holds for all random values of $\mathbf{r}$, and, if $i \in \{2, \ldots, p\}$, even "$>$" holds. Since by either assumption (i) or (ii) of the theorem, the utility values are continuous random variables, the expected value yields

$$E[\,U(\bar{y}_1(\mathbf{r}), \bar{y}_2, \ldots, \bar{y}_p)\,] \;\geq\; E[\,U(\bar{y}_1(\mathbf{r}), y_2, \ldots, y_p)\,]$$

and again "$>$" for $i \in \{2, \ldots, p\}$.

Part (2): Define the following utility function for fixed values of the deterministic objectives

$$\hat{U}(t) := U(t, y_2, \ldots, y_p)\;, \quad t \in R$$

Then it is clear that $E[\,\hat{U}(\bar{y}_1(\mathbf{r}))\,] = E[\,U(\bar{y}_1(\mathbf{r}), y_2, \ldots, y_p)\,]$ and $E[\,\hat{U}(y_1(\mathbf{r}))\,] = E[\,U(y_1(\mathbf{r}), y_2, \ldots, y_p)\,]$. Note that assumptions (i) or (ii) of the Theorem yield Markowitz's original assumptions of $\hat{U}$ being a quadratic utility function or $\mathbf{r}$ being multivariate normally distributed. Since $\bar{\mu} \geq \mu$ and $\bar{\sigma}^2 \leq \sigma^2$, by Markowitz's theorem

$$E[\,\hat{U}(\bar{y}_1(\mathbf{r}))\,] \;\geq\; E[\,\hat{U}(y_1(\mathbf{r}))\,]$$

and even "$>$" if $i \in \{0, 1\}$. □